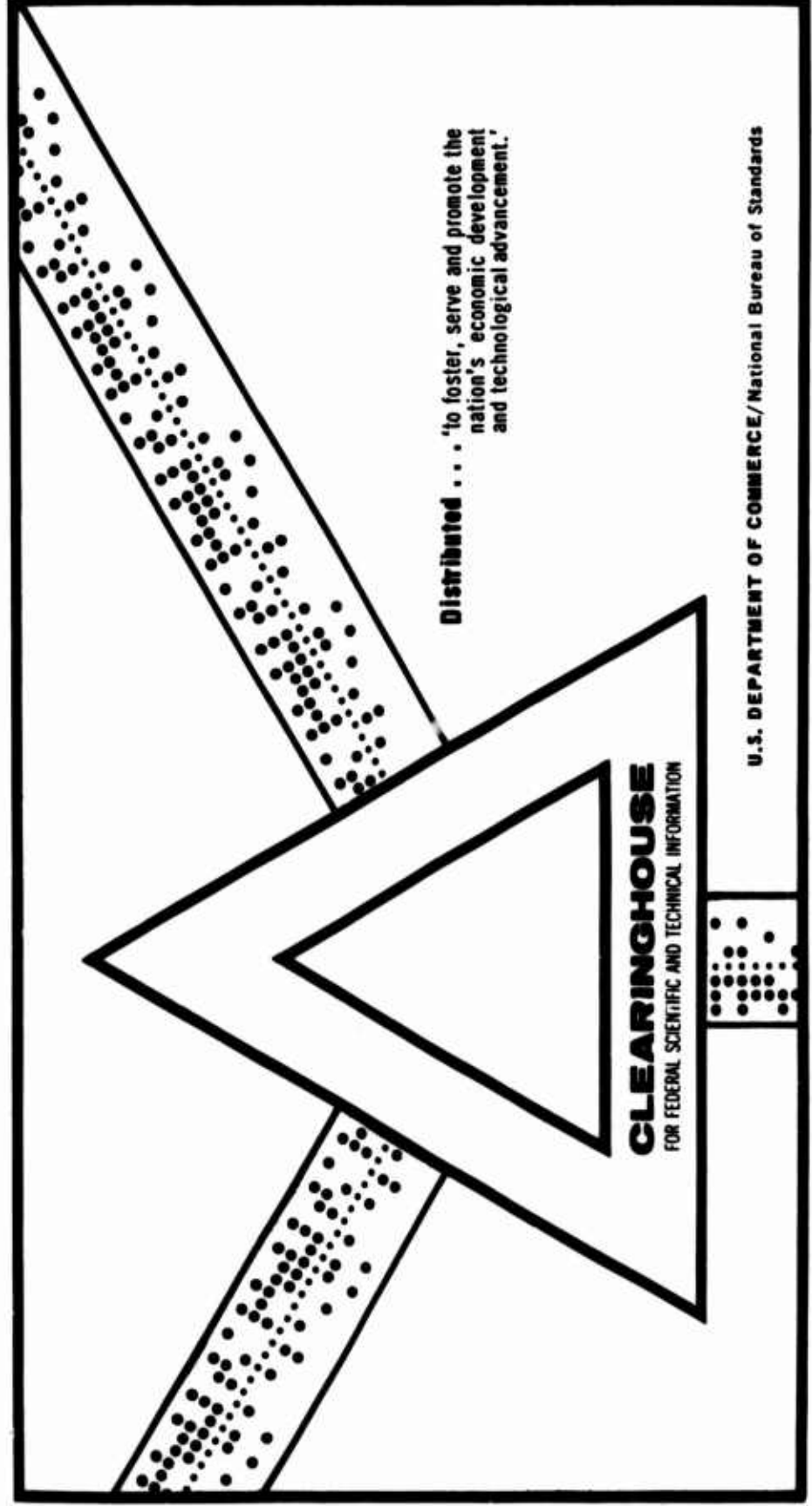AD 701 753

TREE ACCEPTORS AND SOME OF THEIR APPLICATIONS

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ABSTRACT

This paper concerns a generalization of finite automata, the "tree
acceptors," which have as their inputs finite trees of symbols rather than
the usual sequences of symbols. Ordinary finite automata prove to be special
cases of tree acceptors, and many of the results of finite automata theory
continue to hold in their appropriately generalized forms. The tree acceptors
provide new characterizations of the classes of regular sets and of context-
free languages. The theory of tree acceptors is applied to a decision problem
of mathematical logic. It is shown here that the weak second-order theory of
two successors is decidable, thus settling a problem of Buchi. This result is
in turn applied to obtain positive solutions to the decision problems for
various other theories, e.g. the weak second-order theories of order types built
up from the finite types, ω, and η (the type of the rationals) by finitely many
applications of the operations of order type addition, multiplication, and
converse; and the weak second-order theory of locally free algebras with only
unary operations.

TREE ACCEPTORS AND SOME OF THEIR APPLICATIONS[*]

INTRODUCTION

This paper concerns a generalization of a part of finite automata theory. We shall define a generalized finite automaton, called a "tree acceptor," which has as its inputs finite trees of symbols instead of the usual sequences of symbols. Ordinary finite automata prove to be special cases of tree acceptors. It turns out that many of the results of finite automata theory remain valid in their appropriately generalized forms.

Section 1 includes the definitions of trees and tree acceptors, and the development of some of their basic properties. The properties of the sets of trees accepted by tree acceptors are investigated and an alternative characterization of those sets is obtained. An application of the results in Section 1 to the theory of context-free languages is given in Section 2. In Section 3, we give a positive solution to a problem of Buchi [1]: Is the weak second-order theory of two successors decidable? Applications of this result to decision problems of weak second-order logic appear in Section 4; for example, we show that the class of order types with decidable weak second-order theories contains ω, every finite type, and the type of rationals, and is closed under the order-type operations of addition, multiplication, and converse. Finally in Section 5,

a result of Buchi [2] and the generalized products of Feferman and Vaught are utilized to extend the decidability result of Section 3 to a more general case: the weak second-order theory of locally free algebras with only unary operations.

Many of the results in Sections 1 and 3 of this paper were also obtained by J.W. Thatcher and J.B. Wright [28], who use a different, but essentially equivalent, formulation of generalized automata. In fact, Thatcher and Wright were very close to obtaining the decision result in Section 3 when they were notified by Addison (personal communication) of the present author's success. The characterization of context-free languages given in Section 2 is basically that given by J. Mezei and J.B. Wright [19], in a different formulation.

The author wishes to thank Professors J.W. Addison and Alfred Tarski for many stimulating discussions and useful suggestions.

PRELIMINARIES

We shall employ standard set-theoretical notions: $\cap$, $\cup$, $\in$, etc. $A \sim B$ denotes the difference of the sets A, B, i.e., $A \sim B = \{x: x \in A \text{ and } x \notin B\}$. Each ordinal number is defined as the set of all smaller ordinals; 0, the first ordinal, is equal to the empty set $\emptyset$. Thus, the $<$ relation among ordinals coincides with the membership relation $\in$. Finite ordinals 0, 1, 2, ... are called natural numbers and the set of all of them is the first finite ordinal ω. Cardinals are initial ordinals, i.e., ordinals not set-theoretically equivalent to smaller ordinals. The cardinality of a set A is denoted by $\bar{A}$.

If a function f is defined for each element of a class K and $A \subseteq K$, then $f(A) = \{f(x) : x \in A\}$. The domain of a function f is denoted by $\text{dom}(f)$. Assertions of the form "C is the class defined by the conditions..." or "C is the least class such that..." are to be interpreted to mean that C is the intersection of all classes satisfying the stated conditions.

Our notation for automata, words, languages, etc., is, for the most part, adapted from [21] and [14]. An alphabet Σ is a nonempty finite set of symbols (or letters). Unless otherwise stated, the letters Σ, Δ, π, Σ', Δ', ... will denote alphabets. A word over Σ, or simply a word when Σ is understood, is a finite sequence of elements of Σ. A word with only one letter σ is identified with σ itself; ϵ denotes the empty word, and concatenation of words is indicated by juxtaposition. (To facilitate the use of these conventions, we implicitly rule out various "pathological" cases, e.g., we do not admit ϵ as a possible element of an alphabet.) Usually, the small greek letters σ, ξ, μ, ν are used for single elements of an alphabet, and small Roman letters u, v, w, x, y, z for words over an alphabet. The length of a word w is denoted by $|w|$. If A, B are sets of words then $A \cdot B = \{xy : x \in A \text{ and } y \in B\}$. $A^0 = \{\epsilon\}$, and for each finite n, $A^{n+1} = A^n \cdot A$; the union $\bigcup_{n < \omega} A^n$ is denoted by A^*. In particular, if A is an alphabet, then A^* is the set of all words over A.

A set of words A is regular if for some alphabet Σ, A is a member of the last class C such that: (i) every finite subset of Σ^* belongs to C, (ii) C is a Boolean algebra of sets (i.e., if X, $Y \in C$, then $X \cap Y$, $X \cup Y$, $X \sim Y$ are also members of C), (iii) if X, $Y \in C$, then $X \cdot Y \in C$, and (iv) if $X \in C$, then $X^* \in C$.

If Σ is an alphabet, then a Σ-automaton is a 4-tuple $\mathfrak{A} = \langle S, t, s_0, D \rangle$ where S is a nonempty finite set (of states), t is a mapping of $S \times \Sigma$ into S (the transition function), $s_0 \in S$ (the initial state), and $D \subseteq S$ (the designated states). We associate with $\mathfrak{A}$ the function $\bar{t}$, defined recursively: $\bar{t}(\epsilon) = s_0$, and for any $w \in \Sigma^*$ and $\sigma \in \Sigma$, $\bar{t}(w\sigma) = t(\bar{t}(w), \sigma)$. $\mathfrak{A}$ <u>accepts</u> a word $w \in \Sigma^*$ if $\bar{t}(w) \in D$; $T(\mathfrak{A})$ denotes the set of words accepted by $\mathfrak{A}$. We note the well-known result of Kleene:

> A set of words A is regular if and only if $A = T(\mathfrak{A})$
> for some Automaton $\mathfrak{A}$.

Throughout this paper we accept as given a fixed infinite list of distinct letters $a_0, a_1, \ldots$. The alphabets $\{a_0, \ldots, a_p\}$, $p < \omega$, will play a special role. The symbols a_0, a_1, a_2 will also be denoted by a, b, c respectively.

SECTION 1. TREE ACCEPTORS AND RECOGNIZABLE SETS

DEFINITION 1.1. A Σ-tree, or a tree over Σ, of order p, p > 0, is a function $\tau : A \to \Sigma$ where A is a finite subset of $\{a_o, \dots, a_{p-1}\}^*$ closed under the initial segment relation (i.e., if uv $\in$ A, then u $\in$ A).

The small Greek letters τ, π, ρ, τ',... will be used for trees. We adopt the following special notation for trees: The value of a tree τ at a word w $\in$ dom(τ) may be denoted by τ_w as well as $\tau(w)$.

Figure 1 presents graphic representations of two trees over the alphabet $\{\sigma, \zeta, \mu, \nu\}$. In each of the diagrams, the value of the tree at ϵ appears at the apex; below and to the left of the apex is the value at a, below and to the right of this, the value at b, etc. Thus Figure 1(a) is a diagram of the tree τ where $\tau_\epsilon = \sigma$, $\tau_a = \mu$, $\tau_{aa} = \mu$, $\tau_{ab} = \zeta$, $\tau_b = \nu$, $\tau_{bb} = \mu$, and τ is undefined elsewhere. The trees in Figures 1(a) and 1(b) are of orders 2 and 3 respectively.
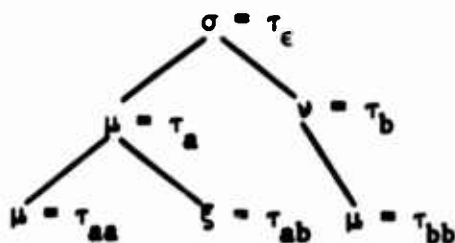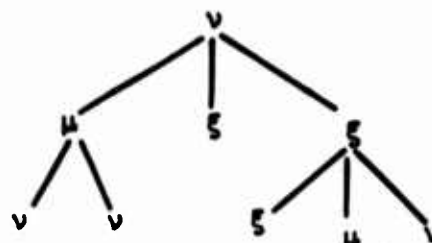
1(a)   1(b)

Figure 1. Two Trees Over the Alphabet $\{\sigma, \zeta, \mu, \nu\}$.

The class of all Σ-trees of a given order p will be denoted by $\Sigma^{\#}$; when we use this notation, the number p will always be either determined by the context or understood to be arbitrary. In most of this section, we shall restrict our consideration to trees of order 2, i.e., trees which are functions with domains which are finite subsets of $\{a,b\}^{*}$. This is done merely for notational convenience; and, usually, the reader will easily be able to supply the rather obvious modifications to our definitions, theorems, and proofs which are required for the transition from order 2 to any finite order p. Following Theorem 1.16, we shall make some further remarks concerning the relationships between sets of trees of various orders.

The empty tree, i.e., the function with domain $\emptyset$, is denoted by Λ. A convention of considerable convenience which we shall adopt is the following: for any tree τ and word w, we write $\tau_{w} = \epsilon$ if and only if $w \notin \text{dom}(\tau)$. Thus, Λ could be defined as the unique tree satisfying the equation $\tau_{e} = \epsilon$. If $\sigma \in \Sigma$, we identify the Σ-tree τ such that $\tau_{\epsilon} = \sigma$ and $\tau_{w} = \epsilon$ for all $w \neq \epsilon$ with the symbol σ itself (of course, σ is also identified with the one letter word σ; nevertheless, no confusion will result from these conventions).

A terminal of a tree τ is a word $w \in \text{dom}(\tau)$ such that no extension of w is also in $\text{dom}(\tau)$. The set of all terminals of τ is called the _frontier_ of τ, denoted by $\text{fr}(\tau)$. The "subtree of τ beginning at w" is $\tau \backslash w$. Formally, if τ is a Σ-tree and $w \in \Sigma^{*}$, then $\tau \backslash w$ is the Σ-tree π such that $\pi_{u} = \tau_{wu}$ for each $\in \{a,b\}^{*}$. It τ, τ' are Σ-trees, then $\tau[w/\tau]$ is the result of replacing the subtree of τ beginning at w with the tree τ', i.e., $\tau[w/\tau']$ is the function π such that

$$\pi_{wv} = \tau'_v \text{ for all } v \in \{a,b\}^*$$

$$\pi_u = \tau_u \text{ for all } u \in \{a,b\}^* \sim (\{w\} \cdot \{a,b\}^*).$$

Notice that $\tau[w/\tau']$ is a Σ-tree only in case $w \in \{ua, ub : u\epsilon \text{ dom}(\tau)\} \cup \{\epsilon\}$. For $\sigma \in \Sigma$ and $\tau, \tau' \in \Sigma^{\#}$, we put $\sigma[\tau,\tau'] = (\sigma[a/\tau]) [b/\tau']$. Thus $\sigma[\tau,\tau']$ is the unique tree π such that $\pi_\epsilon = \sigma$, $\pi\!\!\wedge a = \tau$, and $\pi\!\!\wedge b = \tau'$. Every tree except Λ can be expressed in the form $\sigma[\tau,\tau']$ for some σ, τ, τ'.

The notation $\sigma[\tau,\tau']$ facilitates a form of proof which we call "tree induction"; namely, if for a given proposition $P(\tau)$, where τ ranges over Σ-trees, we can prove

(i) $P(\Lambda)$

(ii)   if $P(\tau)$ and $P(\tau')$, then $P(\sigma[\tau,\tau'])$ for every $\sigma \in \Sigma$,

then we infer $P(\tau)$ for every $\tau \in \Sigma^{\#}$. Corresponding to the principle of tree induction is a form of definition, "tree recursion."

The <u>depth</u> of a tree τ is $\|\tau\| = 1 + n$, where n is the length of the longest word in the domain of τ. An alternative definition of depth is by tree recursion:

$$\| \Lambda \| = 0$$

$$\| \sigma[\tau,\tau'] \| = 1 + \max(\|\tau\|, \|\tau'\|).$$

Proofs by tree induction are, of course, simply inductions on depth; a similar remark applies to definitions by tree recursion.

The concept "Σ-tree" may be regarded as generalizing the concept "Σ-word." The practice of defining sequences as functions of a special kind is common in mathematics; when we construe a Σ-word as a function with range Σ and with domain a finite set consisting of all initial segments of some words in $\{a\}^*$, the generalization to Σ-trees becomes obvious.

Other representations of trees than the one we have given in Definition 1.1 are more common in the literature. Salient among these is the definition of a tree as a partial ordering satisfying certain conditions. This definition does not lend itself to our purposes, since we wish to maintain the distinction between left- and right-branching.

Another approach, quite equivalent to ours, but which we prefer not to adopt here, represents trees as terms in a formal language: The elements of Σ are construed as 2-place function symbols (or p-place function symbols for trees of order p) and a new symbol, λ, which serves as a constant, is introduced. The empty tree Λ is represented by the term λ, and for any $\sigma \in \Sigma$ and τ, $\tau' \in \Sigma^{\#}$, if ψ, ψ' are the terms representing τ, τ' respectively, then $\sigma(\psi, \psi')$ is the term representing $\sigma[\tau, \tau']$. Thus, the tree in Figure 1(a) is represented by the term

$$(1) \quad \sigma(\mu(\mu(\lambda,\lambda), \xi(\lambda,\lambda)), \nu(\lambda,\mu(\lambda,\lambda))).$$

(Notice that the notations we have adopted enable us, in effect, to sometimes make use of the "term representation of trees"; in line (1), we have only to replace the round parentheses (,) by brackets [,] and the symbol λ by Λ to obtain a correct expression for a tree of Figure 1(a).)

The representation of trees as terms in a formal language has its advantages
in certain contexts.   It is essentially the approach used by Thatcher and Wright
in [28]--their "generalized finite automata" have terms as inputs, and using
these, they obtain many results closely related or identical to those which
appear in Sections 1 and 3 of this paper.

Most of the remainder of this section will be devoted to the development of
a generalized notion of finite state acceptors, or finite automata, which admits
trees rather than words as their inputs.   It turns out that a large part of
conventional finite automata theory continues to hold in the generalized context.
Thus, our general approach and most of the theorems and proofs in this section
(and in Section 3 as well) are rather natural adaptations of material found in
the literature on finite automata.   We are particularly indebted to Rabin and
Scott [21], and to Elgot [10].   Occasionally when a proof is very similar to its
corresponding version in one of these papers, we will merely sketch it or omit
it entirely.

<u>DEFINITION 1.2.</u> A Σ-<u>tree acceptor</u> is a 4-tuple $\mathfrak{A} = \langle S,t,s_0,D\rangle$ where

   (i)   S is a nonempty finite set (of <u>states</u>);

   (ii)   t is a mapping of $S \times S \times \Sigma$ into S (the <u>transition function</u>);

   (iii) $s_0 \in S$ (the <u>initial state</u>);

   (iv) $D \subseteq S$ (the set of <u>designated states</u>).

Associated with $\mathfrak{A}$ is the function $\bar{t} : \Sigma^\# \to S$ defined by

$$\bar{t}(\Lambda) = s_0,$$

$$\bar{t}(\sigma[\tau,\tau']) = t(t^-(\tau), \bar{t}(\tau'), \sigma),$$

for all $\sigma \in \Sigma$ and τ, $\tau' \in \Sigma^{\#}$. $\mathfrak{U}$ __accepts__ a tree $\tau \in \Sigma^{\#}$ if $\bar{\mathfrak{t}}(\tau) \in D$. $T(\mathfrak{U})$ denotes the set of Σ-trees accepted by $\mathfrak{U}$.

__DEFINITION 1.3.__ Let $\mathfrak{U} = \langle S, t, s_0, D \rangle$ be a Σ-tree acceptor and let $\tau \in \Sigma^{\#}$. The S-tree π $\mathfrak{U}$-__compatible__ with τ (or simply __compatible__ with τ when $\mathfrak{U}$ is understood) is defined by

   (i) $\mathrm{dom}(\pi) = \{\epsilon\} \cup (\mathrm{dom}(\tau) \cdot \{a, b\})$,

   (ii) $\pi_w = \bar{\mathfrak{t}}(\tau \backslash w)$ for each $w \in \mathrm{dom}(\pi)$.

The tree π compatible with τ might also be called the __state tree__ of τ. Notice that $\|\pi\| = 1 + \|\tau\|$. This is analogous to the situation with finite automata, where a sequence of states compatible with an input word is always one term longer than the word.

__LEMMA 1.4.__ If $\mathfrak{U} = \langle S, t, s_0, D \rangle$ is a Σ-tree acceptor, $\tau \in \Sigma^{\#}$, and π is compatible with τ, then $\tau \in T(\mathfrak{U})$ if and only if $\pi_\epsilon \in D$.

__DEFINITION 1.5__ A set $A \subseteq \Sigma^{\#}$ is __recognizable__ (over Σ) if $A = T(\mathfrak{U})$ for some Σ-tree acceptor $\mathfrak{U}$.[1]

__LEMMA 1.6.__ If Σ_1, Σ_2 are alphabets and $\Sigma_1 \subseteq \Sigma_2$, then a set $A \subseteq \Sigma_1^{\#}$ is recognizable over Σ_1 if and only if A is recognizable over Σ_2.

__THEOREM 1.7.__ The class of recognizable sets is a Boolean algebra; i.e., it is closed under finite unions, finite intersections, and differences.

---

[1]The term __recognizable__ was introduced by Mezei and Wright in [19].

PROOF. Let A, B be two recognizable sets; in view of 1.6, we may assume that they have a common underlying alphabet Σ. Let $\mathfrak{U} = \langle S,t,s_o,D \rangle$ and $\mathfrak{B} = \langle S',t',s_o',D' \rangle$ be Σ-tree acceptors such that $T(\mathfrak{U}) = A$ and $T(\mathfrak{B}) = B$. We shall construct acceptors $\mathfrak{C}, \mathfrak{C}' \ \mathfrak{C}''$ such that $T(\mathfrak{C}) = A \cup B$, $T(\mathfrak{C}') = A \cap B$, and $T(\mathfrak{C}'') = A \sim B$. Let

$$\mathfrak{C} = \langle S \times S', r, \langle s_o, s_o' \rangle, E \rangle$$

where

$$r(\langle s_1, s_1' \rangle, \langle s_2, s_2' \rangle, \sigma) = \langle t(s_1, s_2, \sigma), \ t'(s_1', s_2', \sigma) \rangle$$

for all s_1, $s_2 \in S$, s_1', $s_2' \in S'$, $\sigma \in \Sigma$, and $E = S \times D' \cup D \times S'$. The acceptors $\mathfrak{C}', \mathfrak{C}''$ are obtained from $\mathfrak{C}$ be replacing E by $D \times D'$ and $D \times (s' \sim D')$, respectively. It is easy to verify (e.g., by tree induction) that $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}''$ possess the desired properties.

Note that in the proof of Theorem 1.7, the construction of $\mathfrak{C}, \mathfrak{C}', \mathfrak{C}''$ from the given acceptors $\mathfrak{U}, \mathfrak{B}$ is effective.

The concept of "nondeterministic automata" has proved useful in finite automata theory; although nondeterministic automata are equivalent to ordinary automata with respect to sets of words accepted, they nevertheless are often considerably more convenient to use in the course of proofs. An entirely analogous situation exists in the context of tree acceptors.

DEFINITION 1.8. A nondeterministic Σ-tree acceptor is a 4-tuple $\mathfrak{U} = \langle S,t,I,D \rangle$ where

    (i)   S is a nonempty finite set (of states);

(ii)   t is a mapping of S × S × Σ into the nonempty subsets of S (the

transition function);

(iii)   I is a nonempty subset of S (the initial states);

(iv)   D ⊆ S (the subset of designated states).

When it is necessary to emphasize the distinction, we shall refer to the

tree acceptors of Definition 1.1 as deterministic tree acceptors.

DEFINITION 1.9. Let $\mathfrak{A} = \langle S,t,I,D \rangle$ be a nondeterministic Σ-tree acceptor.

The relation of compatibility between Σ-trees and S-trees is defined by the

following two conditions (i) if $s \in I$, then s is compatible with Λ; (ii) if

π, π' are compatible with τ, τ' respectively, and if $\sigma \in \Sigma$ and $s \in t(\pi_\epsilon, \pi'_\epsilon, \sigma)$,

then s $[\pi, \pi']$ is compatible with $\sigma[\tau, \tau']$. $\mathfrak{A}$ accepts a tree $\tau \in \Sigma^{\#}$ if there

exists an S-tree π compatible with τ such that $\pi_\epsilon \in D$. $T(\mathfrak{A})$ denotes the set of

Σ-trees accepted by $\mathfrak{A}$.

Just as with finite automata, it turns out that the class of sets accepted

by nondeterministic tree acceptors is the same as the class of sets accepted by

deterministic tree acceptors, namely, the recognizable sets. Specifically, by

means of an entirely straightforward generalization of the well-known "subset

construction" used in the proof of Theorem 11 of [21], we obtain

THEOREM 1.10. If $\mathfrak{A}$ is a nondeterministic Σ-tree acceptor, then a deterministic

Σ-tree acceptor $\mathfrak{A}'$ such that $T(\mathfrak{A}) = T(\mathfrak{A}')$ can be effectively obtained.

The following theorem and its corollary are also analogous to corresponding

results of automata theory. Their proofs, however, although based upon ideas

similar to those in the proofs of the corresponding results, do entail some

additional technicalities.

__THEOREM 1.11.__ Let $\mathfrak{A} = \langle S, t, s_o, D \rangle$ be a Σ-tree acceptor. Then $T(\mathfrak{A}) \neq \phi$ if and only if there exists a tree $\tau \in T(\mathfrak{A})$ such that $\|\tau\| < \overline{\overline{S}}$.

__PROOF.__ We need only establish the "only if" part of the equivalence. For any $\tau \in \Sigma^{\#}$, let $n(\tau)$ be the cardinality of the set of $w \in \text{dom}(\tau)$ with $|w| \geq \overline{\overline{S}}$. We wish to show that if $T(\mathfrak{A}) \neq \phi$, then $n(\tau) = 0$ for some $\tau \in T(\mathfrak{A})$. We shall give a procedure which, when applied to any given $\tau \in T(\mathfrak{A})$ such that $n(\tau) > 0$, yields a tree $\tau' \in T(\mathfrak{A})$ with $n(\tau') < n(\tau)$. Applying this procedure finitely many times leads to a tree $\tau'' \in T(\mathfrak{A})$ such that $n(\tau'') = 0$.

Accordingly, let $\tau \in T(\mathfrak{A})$ be such that $n(\tau) > 0$, let w be a terminal of τ such that $|w| \geq \overline{\overline{S}}$, and let π be an S-tree compatible with τ. Now $wa \in \text{dom}(\pi)$ and $|wa| > \overline{\overline{S}}$; hence, there exist words x, y, z such that $y \neq \epsilon$, $wa = xyz$, and $\pi_x = \pi_{xy}$. Let

$$\pi' = \pi[x \,/\, \pi\!\upharpoonright xy],$$
$$\tau' = \tau[x \,/\, \tau\!\upharpoonright xy].$$

Then π' is compatible with τ', $\pi'_\epsilon = \pi_\epsilon$, and hence $\tau' \in T(\mathfrak{A})$. Because $y \neq \epsilon$, we have $w \notin \text{dom}(\tau')$, and since $\text{dom}(\tau') \subseteq \text{dom}(\tau)$, it follows that $n(\tau') < n(\tau)$.

__COROLLARY 1.12.__ If $\mathfrak{A}$ is any tree acceptor, then it is effectively decidable whether

     (i) $T(\mathfrak{A}) = \emptyset$;

     (ii) $T(\mathfrak{A})$ is finite.

PROOF. Part (i) is immediate from Theorem 1.11, since the set $\{\tau : \|\tau\| < \bar{S}\}$ is finite and it is effectively decidable whether $\tau \in T(\mathfrak{U})$. We shall establish part (ii) by showing that $T(\mathfrak{U})$ is infinite if and only if the set

$$A = \{\tau : \|\tau\| \geq \bar{S} \text{ and } \tau \in T(\mathfrak{U})\}$$

is not empty and that it is effectively decidable whether $A = \emptyset$.

Clearly, if $T(\mathfrak{U})$ is infinite, then $A \neq \emptyset$. Now assume that $A \neq \emptyset$. Let $\mathfrak{U}$, τ, π, x, y be as in the proof of Theorem 1.11, and note that $\tau \in A$. We define trees $\tau^{(n)} \in \Sigma^{\#}$, $\pi^{(n)} \in S^{\#}$ for each finite n by recursion:

$$\tau^{(o)} = \tau, \qquad\qquad\qquad \pi^{(o)} = \pi,$$

$$\tau^{(n+1)} = \tau[xy \, / \, \tau^{(n)} \wedge x] \qquad \pi^{(n+1)} = \pi[xy \, / \, \pi^{(n)} \wedge x].$$

Then, for each $n, \pi^{(n)}$ is compatible with $\tau^{(n)}$, $\pi_\epsilon^{(n)} = \pi_\epsilon$, and hence $\tau^{(n)} \in T(\mathfrak{U})$. This shows that $A \neq \emptyset$ implies $T(\mathfrak{U})$ is infinite.

Our demonstration that $A \neq \emptyset$ is effectively decidable involves a modification of the construction in the proof of 1.11. Without sacrificing the essential properties of the procedure given there, we may add the requirement that x be of maximal length in

$$\{x' : \pi_{x'} = \pi_{x'y'} \text{ for some } y', z' \text{ such that } x'y'z' = wa \text{ and } y' \neq \epsilon\}.$$

From this maximality condition on x it follows that $|y| \leq \bar{S}$. Now suppose that the transition from τ to τ' is the last application of the procedure in the proof of 1.11, viz.,

$$\tau, \; \tau' \in T(\mathfrak{A}), \; \|\tau\| \geq \bar{\bar{S}}, \; \tau' = \tau[x \; / \; \tau| \; xy], \; \text{and} \; \|\tau'\| < \bar{\bar{S}}.$$

Then $\|\tau\| = \|\tau'\| + |y|$ and hence $\bar{\bar{S}} \leq \|\tau\| \leq 2 \cdot \bar{\bar{S}}$. Thus, $A \neq \emptyset$ if $A' \neq \emptyset$, where

$$A' = \{\tau \; : \; \tau \in T(\mathfrak{A}) \; \text{and} \; \bar{\bar{S}} \leq \|\tau\| \leq 2 \cdot \bar{\bar{S}}\}.$$

Clearly, it is effectively decidable whether $A' \neq \emptyset$.

Many characterizations of the regular sets are known in the literature. The earliest, due to Kleene, states that a set of words is regular iff it is the set of words accepted by some finite automation. Among the others, we have, for example, that the regular sets coincide with the sets generated by right-linear grammars (Chomsky and Miller [6]), with the sets definable, in a special sense, in a formal language (Buchi [1]; Elgot [10]), and with the sets which are the unions of some of the equivalence classes of a congruence relation of finite index (Myhill [20]). In this paper we shall add two new characterizations of the regular sets to the list; these are Theorem 1.16 and Corellary 3.11.

It seems natural to inquire whether some of the characterizations of the regular sets can be generalized to characterizations of the recognizable sets. This is indeed the case. Thatcher and Wright in their paper [28] give such a generalization of the "·, * characterization" of the regular sets. Here, we shall develop a characterization of the recognizable sets which generalizes Theorem 3.6 of Elgot [10]; many of the ideas involved are closely related to those of Medvedev, [18]. It turns out that this particular characterization is well suited to our later work in Sections 2 and 3.

If Σ is any alphabet and σ, σ', $\sigma'' \in \Sigma \cup \{\epsilon\}$, we denote by $E_\Sigma(\sigma,\sigma',\sigma'')$ the set of all Σ-trees τ such that, for some w, $\tau_w = \sigma$, $\tau_{wa} = \sigma'$, and $\tau_{wb} = \sigma''$. Note that in particular, $E_\Sigma(\epsilon,\epsilon,\epsilon) = \Sigma^\#$, while if one of σ', σ'' is not ϵ, then $E_\Sigma(\epsilon,\sigma',\sigma'') = \emptyset$. For $\sigma \in \Sigma$, the condition that $\tau_w = \sigma$ for some $w \in fr(\tau)$ is expressed simply by $\tau \in E_\Sigma(\sigma,\epsilon,\epsilon)$.

Given two alphabets Σ_1 and Σ_2, we say that a mapping $g : \Sigma_1^\# \to \Sigma_2^\#$ is a <u>projection</u> (of $\Sigma_1^\#$ into $\Sigma_2^\#$) if $g(\Lambda) = \Lambda$ and $(g(\tau))_w = g(\tau_w)$ for all w. (In other words, a projection is the natural extension of a mapping of Σ_1 into Σ_2 to a mapping of $\Sigma_1^\#$ into $\Sigma_2^\#$.) If we are given a mapping of Σ_1 into Σ_2, we speak of the projection defined by this mapping, with the obvious meaning.

Let R be any ternary relation on $\Sigma \cup \{\epsilon\}$. We say that a tree $\tau \in \Sigma^\#$ is <u>R-consistent</u> if $R(\tau_w, \tau_{wa}, \tau_{wb})$ holds for every $w \in \{a,b\}^*$.

<u>DEFINITION 1.13.</u> The class $\mathbb{R}$ is the least class of sets containing each $E_\Sigma(\sigma,\sigma',\sigma'')$ and closed under the Boolean operations (i.e., $\cup, \cap$. and $\sim$) and under arbitrary projections.

<u>LEMMA 1.14.</u> Let Σ be any alphabet, $A \subseteq \Sigma^\#$, and $B \subseteq \Sigma$.

(i)   If R is a ternary relation on $\Sigma \cup \{\epsilon\}$, then the set of R-consistent trees is a member of $\mathbb{R}$.

(ii)   If $\bar{A} < \omega$, then $A \in \mathbb{R}$.

(iii)   If $A \in \mathbb{R}$, then $\{\tau : \tau \in A$ and $\tau_\epsilon \in B\} \in \mathbb{R}$.

(iv)   If $A \in \mathbb{R}$, then $\{\tau : \tau \in A$ and $\tau_w \in B$ for every $w \in fr(\tau)\}$ is a member of $\mathbb{R}$.

PROOF. To prove (i), we let C be the union of the sets $E_\Sigma(\sigma,\sigma'\sigma'')$ such that $R(\sigma,\sigma'\sigma'')$ does not hold, and find that $\Sigma^\# \sim C$ is the set of R-consistent trees. (Of course, $\Sigma^\# \in R$, since, as noted above, $\Sigma^\# = E_\Sigma(\epsilon,\epsilon,\epsilon)$.)

To establish (ii), it suffices to show that $\{\tau\} \in R$ for every $\tau \in \Sigma^\#$. If $\tau = \Lambda$, then $\{\tau\} = \Sigma^\# \sim \bigcup_{\sigma \in \Sigma} E_\Sigma(\sigma,\epsilon,\epsilon)$. Now suppose $\tau \neq \Lambda$. For each $w \in \text{dom}(\tau)$, let $\zeta^{(w)}$ be a distinct new symbol, and put $\zeta^{(w)} = \epsilon$ for each $w \notin \text{dom}(\tau)$. We let $\Pi = \{\zeta^{(w)} : w \in \text{dom}(\tau)\}$, and defines the relation R on $\Pi \cup \{\epsilon\}$ by $R(\zeta^{(w)},\zeta^{(wa)},\zeta^{(wb)})$ for every w. There is just one R-consistent tree $\pi \in \Pi^\#$, and, by (i), $\{\pi\} \in R$. Let $g : \Pi^\# \to \Sigma^\#$ be the projection defined by $g(\zeta^{(w)}) = \tau_w$ for each $w \in \text{dom}(\tau)$. Then $g(\{\pi\}) = \{\tau\}$, so $\{\tau\} \in R$.

Next, assume that $\Lambda \in R$. Let ζ be a symbol not in Σ, and put $\Sigma' = \Sigma \cup \{\zeta\}$. Let $R = (\Sigma' \cup \{\epsilon\}) \times (\Sigma \cup \{\epsilon\}) \times (\Sigma \cup \{\epsilon\})$, and let C be the set of R-consistent Σ'-trees. Then $C \in R$ by (i), and for $\tau \in C$ we have $\tau_w = \zeta$ only in case $w = \epsilon$. For each $\sigma \in \Sigma$, p_σ is the projection of $\Sigma^\#$ into $\Sigma^\#$ defined by $p_\sigma(\zeta) = \sigma$ and $p_\sigma(\mu) = \mu$ for all $\mu \in \Sigma$. We then have

$$\{\tau : \tau \in A \text{ and } \tau_\epsilon \in B\} = \bigcup_{\sigma \in B}[p_\sigma(C \sim \Sigma^\#) \cap A].$$

This proves (iii).

Finally, to establish (iv) we merely note that

$$A \cap [\Sigma^\# \sim \bigcup_{\sigma \notin B \cup \{\epsilon\}} E_\Sigma(\sigma,\epsilon,\epsilon)] = \{\tau : \tau \in A \text{ and } \tau_w \in B \text{ for every } w \in \text{fr}(\tau)\}.$$

THEOREM 1.15. The class R coincides with the class of recognizable sets.

PROOF. We begin by showing that $\mathbf{R}$ contains every recognizable set. Let $\mathfrak{U} = \langle S, t, s_0, D \rangle$ by any Σ-tree acceptor. Putting $\Pi = S \times \Sigma$, we let R be the ternary relation on $\Pi \cup \{\epsilon\}$ such that, for s, s', $s'' \in S$ and σ, σ', $\sigma'' \in \Sigma$,

$$R(\langle s,\sigma\rangle, \langle s',\sigma'\rangle, \langle s'',\sigma''\rangle) \text{ iff } t(s',s'',\sigma) = s,$$

$$R(\langle s,\sigma\rangle, \epsilon, \langle s'',\sigma''\rangle) \text{ iff } t(s_0,s'',\sigma) = s,$$

$$R(\langle s,\sigma\rangle, \langle s',\sigma'\rangle, \epsilon) \text{ iff } t(s',s_0,\sigma) = s,$$

and

$$R(\langle s,\sigma\rangle, \epsilon, \epsilon) \text{ iff } t(s_0,s_0,\sigma) = s.$$

Let C be the set of R-consistent trees; then $C \in \mathbf{R}$ by 1.14 (i). Let $p_0 : \Pi^\# \to S^\#$, $p_1 : \Pi^\# \to \Sigma^\#$ be the projections such that, for any $\langle s,\sigma\rangle \in \pi$, $p_0(\langle s,\sigma\rangle) = s$ and $p_1(\langle s,\sigma\rangle) = \sigma$. Now for any $\rho \in S^\#$, let $f(\rho)$ be the tree ρ' such that $\text{dom}(\rho') = \{\epsilon\} \cup \{wa, wb : w \in \text{dom}(\rho)\}$ and $\rho'_u = s_0$ for each $u \in \text{dom}(\rho') \sim \text{dom}(\rho)$. The following three propositions can now be proved simultaneously by tree induction:

(i) $p_1(C) = \Sigma^\#$,

(ii)   if $\pi \in C$, then $\pi \mathbin{\wedge} w \in C$ for any w,

(iii) $f(p_0(\pi))$ is $\mathfrak{U}$-compatible with $p_1(\pi) \in C$;

We omit the tedious but entirely routine argument required. The subset C' of C consisting of those trees π such that $f(p_0(\pi))_\epsilon \in D$ (i.e., such that $\pi_\epsilon \in D \times \Sigma$, or $\pi = \Lambda$ in case $s_0 \in D$) is a member of $\mathbf{R}$ by 1.14 (iii); we then have $p_1(c') = T(\mathfrak{U})$, and hence $t(\mathfrak{U}) \in \mathbf{R}$.

To complete the proof of Theorem 1.15, we must show that every element of R is recognizable. In view of Theorem 1.7, it suffices to show that each $E_\Sigma(\sigma,\sigma',\sigma'')$ is recognizable, and that the projection of a recognizable set is again a recognizable set.

Let Σ be an alphabet, and σ, σ', $\sigma'' \in \Sigma \cup \{\epsilon\}$. We first assume that $\sigma \neq \epsilon$. For each $\mu \in \Sigma \cup \{\epsilon\}$, let s_μ be a distinct new symbol, let s_D be another symbol not among these, and put $S = \{s_D\} \cup \{s_\mu : \mu \in \Sigma \cup \{\epsilon\}\}$. The function $t : S \times S \times \Sigma \to S$ is defined as follows: for $\mu \in \Sigma$, μ', $\mu'' \in \Sigma \cup \{\epsilon\}$,

$$t(s_{\mu'},s_{\mu''},\mu) = s_D \text{ if } \mu = \sigma, \ \mu' = \sigma', \text{ and } \mu'' = \sigma'',$$

$$= s_\mu \text{ otherwise,}$$

and

$$t(s_{\mu'},s_D,\mu) = t(s_D,s_D,\mu) = s_D.$$

Putting $\mathfrak{U} = \langle S,t,s_\epsilon,\{s_D\}\rangle$ we have that $\mathfrak{U}$ is a Σ-tree acceptor and $T(\mathfrak{U}) = E_\Sigma(\sigma,\sigma',\sigma'')$; thus, $E_\Sigma(\sigma,\sigma',\sigma'')$ is recognizable whenever $\sigma \neq \epsilon$. In case $\sigma = \epsilon$, then $E_\Sigma(\sigma,\sigma',\sigma'')$ is either $\Sigma^\#$ or $\emptyset$; both of these are recognizable sets, since if $\mathfrak{U} = \langle S,t,s_0,D\rangle$ is any Σ-tree acceptor such that $D = S$, then $T(\mathfrak{U}) = \Sigma^\#$, whereas if $D = \emptyset$, the $T(\mathfrak{U}) = \emptyset$.

Finally, we assume that Σ_1, Σ_2 are two alphabets and that g is a projection of $\Sigma_1^\#$ into $\Sigma_2^\#$. We wish to show that if $A \subseteq \Sigma_1^\#$ is recognizable, then $g(A)$ is recognizable. Let $\mathfrak{U} = \langle S,t,s_0,D\rangle$ be a Σ-tree acceptor such that $T(\mathfrak{U}) = A$. Without loss of generality, we may assume that g maps Σ_1 into Σ_2. Let $\mathfrak{B} = \langle S,t',\{s_0\},D\rangle$ where

$t'(s,s',\sigma) = \{s'' : s'' = t(s,s',\mu)$ for some μ such that $g(\mu) = \sigma\}$,

for each $\sigma \in \Sigma_2$. $\mathcal{B}$ is a nondeterministic Σ_2-tree acceptor. A straightforward argument by tree induction shows that an S-tree π is $\mathcal{B}$-compatible with a Σ_2-tree τ iff π is $\mathcal{U}$-compatible with some Σ_1-tree τ' such that $g(\tau') = \tau$. From this it follows that $g(A) = T(\mathcal{B})$ and hence that $g(A)$ is recognizable.

As a consequence of Theorem 1.15, we have that all the properties of $\mathcal{R}$ given in Definition 1.13 and Theorem 1.14 apply to the class of recognizable sets. We shall often make use of this fact without explicitly citing 1.13, 1.14, and 1.15.

THEOREM 1.16. A set $X \subseteq \{a,b\}^*$ is regular if and only if $X = \bigcup_{\tau \in A} fr(\tau)$ for some recognizable set A.

PROOF. Assume that X is regular and let $\mathcal{U} = \langle S, t, s_o, D \rangle$ be a $\{a,b\}$-automaton such that $T(\mathcal{U}) = X$. Let J be the subset of $S \times (S \cup \{\epsilon\}) \times (S \cup \{\epsilon\})$ such that $\langle s, s', s'' \rangle \in J$ iff either $s' = \epsilon$ and $s'' = t(s,b)$, or else $s'' = \epsilon$ and $s' = t(s,a)$. Let A' be the set of J-consistent S-trees τ such that $\tau_\epsilon = s_o$; A' is recognizable by 1.14 (i), (iii). A simple argument by induction shows that every $w \in \{a,b\}^*$ is a terminal of some member of A'. Now suppose $\tau \in A'$ and $w \in dom(\tau)$. We shall prove by induction on $|w|$ that $\bar{t}(w) = \tau_w$. If $w = \epsilon$, the $\bar{t}(w) = s_o = \tau_\epsilon$. If $|w| > 0$, say $w = ua$, and $\bar{t}(u) = \tau_u$, then, by the J-consistency of τ, we have $\tau_{ua} = t(\tau_u, a)$, and hence, $\bar{t}(w) = t(\bar{t}(u), a) = \tau_w$. Now let $A = \{\tau : \tau \in A$ and $\tau_w \in D$ for $w \in fr(\tau)\}$; then A is recognizable, and from the remarks above, $\bar{t}(w) \in D$ iff $w \in fr(\tau)$ for some $\tau \in A$. It follows that $\bigcup_{\tau \in A} fr(\tau) = T(\mathcal{U}) = X$.

Conversely, assume that A is recognizable and $X = \bigcup_{\tau \in A} fr(\tau)$. Let $\mathfrak{U} = \langle S, t, s_0, D \rangle$ be a Σ-tree acceptor such that $T(\mathfrak{U}) = A$. We define a sequence of sets D_w, $w \in \{a,b\}^*$, as follows:

$$D_\epsilon = D,$$

$$D_{wa} = \{s : t(s, \bar{t}(\tau), \sigma) \in D_w \text{ for some } \tau \in \Sigma^\# \text{ and } \sigma \in \Sigma\},$$

$$D_{wb} = \{s : t(\bar{t}(\tau), s, \sigma) \in D_w \text{ for some } \tau \in \Sigma^\# \text{ and } \sigma \in \Sigma\}.$$

Now let $\mathfrak{U}_w = \langle S, t, s_0, D_w \rangle$ for each $w \in \{a,b\}^*$. Then $T(\mathfrak{U}_w) = \{\tau \backslash w : \tau \in T(\mathfrak{U})\}$, so that $w \in \bigcup_{\tau \in A} fr(\tau)$ iff $\Sigma \cap T(\mathfrak{U}_w) \neq \emptyset$. Let $\mathfrak{B} = \langle B, r, D, F \rangle$ be a $\{a,b\}$-automaton, where

$$B = \{S' : S' \subseteq S \text{ and } S' \neq \emptyset\},$$

and for each $S' \in B$,

$$r(S',a) = \{s : t(s, \bar{t}(\tau), \sigma) \in S' \text{ for some } \tau \in \Sigma^\# \text{ and } \sigma \in \Sigma\},$$

$$r(S',b) = \{s : t(\bar{t}(\tau), s, \sigma) \in S \text{ for some } \tau \in \Sigma^\# \text{ and } \sigma \in \Sigma\},$$

and finally,

$$F = \{S' : S' \subseteq S \text{ and } t(s_0, s_0, \sigma) \in S' \text{ for some } \sigma \in \Sigma\}.$$

Let $w \in \{a,b\}^*$; it follows, by induction on $|w|$, that $\bar{r}(w) = D_w$, and since $w \in T(\mathfrak{B})$ iff $t(s_0, s_0, \sigma) \in \bar{r}(w)$ for some $\sigma \in \Sigma$, we then have that $w \in T(\mathfrak{B})$ iff $\Sigma \cap T(\mathfrak{U}_w) \neq \emptyset$, i.e., iff $w \in \bigcup_{\tau \in A} fr(\tau)$.

The construction of the automaton $\mathfrak{B}$ from the given tree acceptor $\mathfrak{U}$ in the proof of Theorem 1.16 may be made effective; we need merely note that there exists a tree τ such that $\bar{t}(\tau) = s$ iff there exists such a tree depth $< \bar{\bar{s}}$.

Except in Definition 1.1, we have so far restricted consideration to trees of order 2. The modifications to our development required to effect the transition to trees of any finite order $p > 0$ are entirely straightforward: for example, the notation $\sigma[\tau,\tau']$ is changed to, for any $n < p$,

$$\sigma[\tau,\tau',\ldots,\tau^{(p)}] = (\ldots(\sigma[a_0 / \tau])[a_1 / \tau'])\ldots[a_{p-1} / \tau^{(p)}];$$

in Definition 1.2, the transition function t has domain $S^{(p)} \times \Sigma$ instead of $S^{(2)} \times \Sigma$ (where $S^{(1)} = S$ and $S^{(n+1)} = S^{(n)} \times S$); and in Definition 1.13, we replace $E_\Sigma(\sigma,\sigma',\sigma'')$ by

$$E_\Sigma(\sigma,\sigma_0,\ldots,\sigma_{p-1}) = \{\tau : \text{for some } w,\ \tau_w = \sigma,\ \tau_{wa_0} = \sigma_0,\ldots,\ \tau_{wa_p} = \sigma_{p-1}\}.$$

With these modifications, we can extend our concept of "recognizable set" to apply to sets of trees of any given finite order p.

In the remaining sections of this paper, we shall assume that these modifications have actually been carried out. Thus, we shall speak of tree acceptors of order p and recognizable sets of order p, and we shall cite theorems of Section 1 with the understanding that, if necessary, they are to be modified to apply to trees, acceptors, etc., of arbitrary finite orders.

A Σ-tree of order 1 is essentially the same as a finite sequence of members of Σ, i.e., a Σ-word. Consequently, one may identify tree acceptors of order 1 with ordinary finite automata and the recognizable sets of order 1 with the regular sets, so that automata theory becomes a special case of the theory of tree acceptors.

It is an easy consequence of Definition 1.1 that a tree τ of order p is also of order p' for any p' > p. We may naturally inquire whether a recognizable set of trees of order p remains a recognizable set when it is regarded as a set of trees of order p' > p. This is indeed the case; in fact, by simple constructions of tree acceptors we obtain

LEMMA 1.17. Let $\mathfrak{U}$ be a tree acceptor or order p > 0.

- (i)   If p' > p, then there is a tree acceptor $\mathfrak{U}'$ of order p' such that $T(\mathfrak{U}) = T(\mathfrak{U}')$.

- (ii)   If p > p' > 0 and every tree in $T(\mathfrak{U})$ is of order p', then there is a tree acceptor $\mathfrak{U}'$ of order p' such that $T(\mathfrak{U}) = T(\mathfrak{U}')$.

- (iii) If p' > p > 0, then a set A of trees is a recognizable set of order p if and only if A is a recognizable set of order p'.

As a consequence of Lemma 1.17, we have that Theorem 1.7 holds even if no restriction is placed upon the order of the recognizable sets involved. Lemma 1.17 (iii) states, roughly speaking, that recognizability is a property independent of order, so that we may describe a set as recognizable without specifying its order.

Notice that Theorem 1.16 may now be improved as follows: It places no essential restriction on the regular sets to assume that their underlying alphabets are always subsets of $\{a_i : i = 0, 1, \ldots\}$, and under this assumption we have that

A set X is regular if and only if $X = \bigcup_{\tau \in A} fr(\tau)$ for some recognizable set A.

SECTION 2. A CHARACTERIZATION OF CONTEXT-FREE LANGUAGES

In this section we shall give an example of the application of the results of Section 1 to the theory of algorithmic languages; namely, we shall characterize the context-free languages by means of recognizable sets. These results were first obtained by Mezei and Wright [19], although their formulation is technically different from ours. Ginsburg [14] is our principal source for notation, terminology, and results concerning context-free languages.

A <u>context-free grammar</u> is a 4-tuple $G = \langle V, \Sigma, P, \mu \rangle$ where V and Σ are alphabets, $\Sigma \subseteq V$, P is a finite subset of $(V-\Sigma) \times V^*$, and $\mu \in V$. Elements of $V-\Sigma$ are called <u>variables</u>, elements of Σ are <u>constants</u> and elements of P are called <u>productions</u>; a production $\langle \zeta, v \rangle \in P$ is denoted by $\zeta \to v$. For $u, v \in V^*$, we write $u \Rightarrow_G v$ (or simply $u \Rightarrow v$ when G is understood) if for some $u_0, u_1, v' \in V^*$, and $\zeta \in V$, we have $u = u_0 \zeta u_1$, $\zeta \to v'$, and $v = u_0 v' u_1$. We write $u \Rightarrow_G^* v$ if there exists a finite sequence of words $u_0, \ldots, u_n \in V^*$ such that $u = u_0, u_n = v$, and for each $i < n$, $u_1 \Rightarrow_G u_{i+1}$; the sequence $u_0, \ldots, u_n$ is then called a <u>derivation</u> of v from u. The <u>language generated</u> by G, $L(G)$, is the set of words $w \in \Sigma^*$ such that $\mu \Rightarrow^* w$. Of course, if $\mu \in \Sigma$, then $L(g) = \{\mu\}$. A set of L is a <u>context-free language</u> if $L = L(G)$ for some context-free grammar G.

A grammar $G = \langle V, \Sigma, P, \mu \rangle$ is called <u>ϵ-free</u> if it has no production of the form $\zeta \to \epsilon$. Theorem 1.8.1 of [14] states that for any context-free language L there exists a ϵ-free grammar G such that $L(G) = L \sim \{\epsilon\}$. Another result we require from the theory of context-free languages (c.f. [14], Lemma 1.4.6) is the following: for any $\zeta \in V - \Sigma$, and $u \in V^*$, $\zeta \Rightarrow^* u$ iff either $\zeta = u$, or there

is a production $\xi \to \zeta_0 \cdots \zeta_{n-1}$ in P, and words $u_0, \ldots, u_{n-1} \in V^*$ such that for $i < n$, $\zeta_i \Rightarrow^* u_i$, and $u = u_0 \cdots u_{n-1}$.

Given an ϵ-free grammar $G = \langle V, \Sigma, P, \mu \rangle$, V-trees can be associated in a natural way with derivations $\xi = u_0 \Rightarrow \cdots \Rightarrow u_{n-1}$, where $\xi \in V$. In fact whenever $\xi \Rightarrow^* u \in V^*$, there is at least one such V-tree τ such that $\tau_\epsilon = \xi$, and $u = \tau_{w_0} \cdots \tau_{w_n}$, where $w_0, \ldots, w_n$ are the terminals of τ in lexicographical order. The formal details of the correspondence between V-trees and derivations are set forth in the following definitions and lemma.

<u>DEFINITION 2.1.</u> The operator Q on arbitrary trees is defined by tree recursion:

    (i) $Q(\Lambda) = \epsilon$,

    (ii) $Q(\sigma[\tau^{(0)}, \ldots, \tau^{(p)}]) = \sigma$ if $\tau^{(0)} = \ldots = \tau^{(p)} = \Lambda$,

$$= Q(\tau^{(0)}) \ldots Q(\tau^{(p)}) \text{ otherwise.}$$

$Q(\tau)$ is simply the concatenation of the symbols appearing at the terminals of τ, taking the terminals of τ in lexicographical order.

<u>DEFINITION 2.2.</u> Let $G = \langle V, \Sigma, P, \mu \rangle$ be an ϵ-free grammar. C_G is the set of V-trees defined by the conditions

    (i) $V \subseteq C_G$,

    (ii)   if $\tau^{(0)}, \ldots, \tau^{(n)} \in C_G$, and $\xi \to \tau_\epsilon^{(0)} \ldots \tau_\epsilon^{(n)}$, then $\xi[\tau^{(0)}, \ldots, \tau^{(n)}] \in C_G$.

The order of the trees in C_G is the maximum of the set

$$\{1\} \cup \{|u| : \xi \to u \text{ is in P for some } \xi \in V\}.$$

LEMMA 2.3. Let $G = \langle V, \Sigma, P, \mu \rangle$ be an ϵ-free grammar, $\xi \in V$ and $u \in V^*$. Then $\xi \Rightarrow^* u$ iff there is a $\tau \in C_G$ such that $\tau_\epsilon = \xi$ and $Q(\tau) = u$.

Proof. First, assume $\xi \Rightarrow^* u$. We proceed by induction on the length of the derivation establishing $\xi \Rightarrow^* u$. If this length is 0, then $\xi = u$ and $\xi \in C_G$. Otherwise, there are $\zeta_o, \ldots, \zeta_n \in V$ and $u_o, \ldots, u_n \in V^*$ such that $\xi \to \zeta_o \cdots \zeta_n, u = u_o \ldots u_n$, and for each $i, \zeta_i \Rightarrow^* u_i$ by means of a shorter derivation. Applying the inductive hypothesis, we obtain trees $\tau^{(i)} \in C_G$ such that $\tau_\epsilon^{(i)} = \zeta_i$ and $Q(\tau)^{(i)}) = u_i$. Putting $\tau = \xi[\tau^{(o)}, \ldots, \tau^{(n)}]$ we find $\tau \in C_G$ and $Q(\tau) = u$.

Conversely, assume $\tau \in C_G$, $\tau_\epsilon = \xi$, and $Q(\tau) = u$. We proceed by tree induction. $\tau = \Lambda$ is impossible. If $\|\tau\| = 1$, then $\tau = \xi = Q(\tau)$, and $\xi \Rightarrow^* \xi$. If $\|\tau\| > 1$, then $\tau = \xi[\tau^{(o)}, \ldots, \tau^{(n)}]$ for some $\tau^{(o)}, \ldots, \tau^{(n)} \in C_G$ and ξ such that $\xi \to \tau_\epsilon^{(o)} \ldots \tau_\epsilon^{(n)}$. No $\tau^{(i)}$ is Λ, so $Q(\tau) = Q(\tau^{(o)}) \ldots Q(\tau^{(n)})$. But by the inductive hypothesis, $\tau_\epsilon^{(i)} \Rightarrow^* Q(\tau^{(i)})$, for $i \leq n$. Hence, $\xi \Rightarrow^* Q(\tau^{(o)}) \ldots Q(\tau^{(n)}) = Q(\tau) = u$.

The set of trees $\tau \in C_G$ with $\tau_\epsilon = u$ and $Q(\tau) \in \Sigma^*$, where $G = \langle V, \Sigma, P, \mu \rangle$ is a ϵ-free grammar, is simply the set of "derivation trees" for G, a concept well-known in the literature. This set will be denoted by A_G. The impact of Lemma 2.3 is simply that $L(G) = Q(A_G)$.

LEMMA 2.4. If L is a context-free language, then $L = Q(A)$ for some recognizable set A.

PROOF. Let G be a grammar such that $L \sim \{\epsilon\} = L(G)$. We will show that A_G is recognizable. Let p be the length of the longest word u which occurs in a production $\xi \to u$ in P. The $p + 1$ place relation R on $\Sigma \cup \{\epsilon\}$ is defined as

follows: for any ξ, ζ_0, ..., ζ_m, $m < p$, $R(\xi,\zeta_0,...,\zeta_m,...,\epsilon)$ if and only if $\xi \to \zeta_0 \cdots \zeta_m$ is in P. Then A_G is the set of R-consistent trees τ such that $\tau_\epsilon = \mu$ and $\tau_w \in \Sigma$ for every w $fr(\tau)$. Thus, A_G is a recognizable set. The desired result now follows, since we have either $L = Q(A_G)$ or $L = Q(A_G \cup \{\Lambda\})$ according as $\epsilon \notin L$ or $\epsilon \in L$, while both A_G and $A_G \cup \{\Lambda\}$ are recognizable.

Lemma 2.4 may come as no surprise to those familiar with the theory of context-free languages. Somewhat less obvious is the fact that the converse of 2.4 also holds--that $Q(A)$ is a context-free language whenever A is a recognizable set.

THEOREM 2.5. A set L is a context-free language if and only $L = Q(A)$ for some recognizable set A.

PROOF. The "only if" part has already been established as Lemma 2.4. We shall show that if $\mathfrak{A} = \langle S,t,s_0,D \rangle$ is a tree acceptor (of order p) then $Q(T(\mathfrak{A}))$ is a context-free language.[2] Let μ be a new symbol not in Σ or $S \times (\Sigma \cup \{\epsilon\})$, and let G be the context-free grammar

$$G = \langle \Sigma \cup (S \times (\Sigma \cup \{\epsilon\})),\Sigma,P,\mu \rangle$$

where P contains the following productions:

$$\mu \to \langle r,\delta \rangle \text{ for some } r \in D \text{ and } \delta \in \Sigma \cup \{\epsilon\},$$
$$\langle s_0,\epsilon \rangle \to \epsilon,$$

_____

[2]The author would like to express his thanks to the referee for suggesting this proof, which is considerably simpler than the original.

$$\langle t(s_0, \ldots, s_0, \sigma), \sigma \rangle \rightarrow \text{ for all } \sigma \in \Sigma,$$

$$\langle t(s_1, \ldots, s_p, \sigma), \sigma \rangle \rightarrow \langle s, \delta_1 \rangle \ldots \langle s_p, \delta_p \rangle, \text{ for all } s_1 \in \Sigma,$$

$$\text{and } \delta_1 \in \Sigma \cup \{\epsilon\} \text{ with } \delta_1 \neq \epsilon \text{ for at}$$
$$\text{least one i.}$$

First we shall prove

(1)　$\langle \bar{t}(\tau), \tau_\epsilon \rangle \Rightarrow^* Q(\tau)$ for all $\tau \in \Sigma$

by tree induction. If $\tau = \Lambda$, we have $\bar{t}(\tau) = s_0$, $\tau_\epsilon = \epsilon$, $Q(\tau) = \epsilon$, and need merely note that $\langle s_0, \epsilon \rangle \rightarrow \epsilon$. If $\tau = \sigma \in \Sigma$, then $\bar{t}(\tau) = t(s_0, \ldots, s_0, \sigma)$, and $\tau_\epsilon = Q(\tau) = \sigma$. Then $\langle t(s_0, \ldots, s_0, \sigma), \sigma \rangle \rightarrow Q(\tau)$ by definition of Γ. Finally, suppose $\tau = \sigma[\tau^{(1)}, \ldots, \tau^{(p)}]$ with at least one $\tau^{(1)} \neq \Lambda$. By the inductive hypothesis, $\langle \bar{t}(\tau^{(1)}), \tau_\epsilon^{(1)} \rangle \Rightarrow^* Q(\tau^{(1)})$, for $i = 1, \ldots, p$. Now $\bar{t}(\tau) = t(\bar{t}(\tau^{(1)}), \ldots, \bar{t}(\tau^{(p)}), \sigma)$, and since at least one $\tau^{(1)}$ is not Λ, we have

$$\langle \bar{t}(\tau), \sigma \rangle \rightarrow \langle \bar{t}(\tau^{(1)}), \tau_\epsilon^{(1)} \rangle \ldots \langle \bar{t}(\tau^{(p)}), \tau_\epsilon^{(p)} \rangle$$

by definition of P. Since $Q(\tau) = Q(\tau^{(1)}) \ldots Q(\tau^{(p)})$, we have shown $\langle \bar{t}(\tau), \sigma \rangle \Rightarrow^* Q(\tau)$.

Next we shall prove

(2) If $\langle s, \sigma \rangle \Rightarrow^* u \in \Sigma^*$, then $u = Q(\tau)$ for some τ with $\bar{t}(\tau) = s$ and $\tau_\epsilon = \sigma$. This will be done by induction on the length n of the shortest derivation establishing $\langle s, \sigma \rangle \Rightarrow^* u$. $n = 0$ is impossible since $\langle s, \sigma \rangle \notin \Sigma^*$. If $n = 1$ then the only possible production is $\langle t(s_0, \ldots, s_0, \sigma), \sigma \rangle \rightarrow \sigma$, so we must have $u = \sigma$ and $s = t(s_0, \ldots, s_0, \sigma)$. We merely take $\tau = \sigma$. If $n = 1$ and $u = \epsilon$ then the only possible production is $\langle s_0, \epsilon \rangle \rightarrow \epsilon$, so take $\tau = \Lambda$. Finally, suppose $n > 1$ and (2) holds whenever the underlying derivation has fewer than

n steps. There is a production $\langle s,\sigma \rangle \rightarrow \langle s_1,\delta_1 \rangle \ldots \langle s_p,\delta_p \rangle$ and words $u_1, \ldots,$ $u_p \in \Sigma^*$ such that $\langle s_1,\delta_1 \rangle \Rightarrow^* u_1$ (with derivations shorter than n) and $u = u_1 \ldots u_p$. By definition of P, $s = t(s_1,\ldots,s_p,\sigma)$, and by the inductive hypothesis, there are trees $\tau^{(i)}$, $i = 1, \ldots, p$, such that $\bar{t}(\tau^{(i)}) = s_i$, $\tau_\epsilon^{(i)} = \delta_i$, and $Q(\tau^{(i)}) = u_i$. Let $\tau = \sigma[\tau^{(1)},\ldots,\tau^{(p)}]$. Then $\bar{t}(\tau) = s$ and $\tau_\epsilon = \sigma$. Since at least one δ_i is not ϵ, at least one $\tau^{(i)}$ is not Λ; hence, $Q(\tau) = Q(\tau^{(1)})\ldots Q(\tau^{(p)}) = u$. This completes the proof of (2).

Suppose $\tau \in T(\mathfrak{A})$. By (1), $\langle \bar{t}(\tau),\tau_\epsilon \rangle \Rightarrow^* Q(\tau)$. Since $\tau \in T(\mathfrak{A})$, $\bar{t}(\tau) \in D$, so $\mu \rightarrow \langle \bar{t}(\tau),\tau_\epsilon \rangle$. Then $\mu \Rightarrow^* Q(\tau)$, and it follows that $Q(T(\mathfrak{A})) \subseteq L(G)$. Conversely, suppose $u \in L(G)$. For some $s \in D$, $\delta \in \Sigma \cup \{\epsilon\}$, $\mu \rightarrow \langle s,\delta \rangle$ and $\langle s,\delta \rangle \Rightarrow^* u$. We have from (2) that there is a tree τ such that $Q(\tau) = u$, $\bar{t}(\tau) = s$, and $\tau_\epsilon = \delta$. Since $s \in D$, this means that $\tau \in T(\mathfrak{A})$, and it follows that $L(G) \subseteq Q(T(\mathfrak{A}))$.

SECTION 3. DECIDABILITY OF THE THEORY OF p SUCCESSOR OPERATIONS

In this section, we apply the theory of recognizable sets to a decision
problem of mathematical logic: We will show that, for any $p < \omega$, the weak
second·order theory of p successor operations is decidable (Corollary 3.8).
This answers in the affirmative a problem of Buchi, stated in Section 9 of
[1]. In case $p = 1$, this result was first reported by Buchi and Elgot [5], and
published by them in [1] and [10]. Most of the methods employed in this section
are generalizations of those used by Elgot in [10].

Let B be any set; let p, q be any ordinals; for $i < p$, let O_i be a m_i-ary
operation on B; and for $j < q$ let R_j be a n_j-ary relation among the elements of
B. Then we say that the system

$$\mathbf{B} = \langle B, O_0, \ldots, O_i, \ldots, R_0, \ldots, R_j, \ldots \rangle_{i<p,\, j<q}$$

in a <u>algebraic structure of similarity type</u> $\alpha = \langle \langle m_0, \ldots, m_i, \ldots \rangle_{i<p},$
$\langle n_0, \ldots, n_j, \ldots \rangle_{j<q} \rangle$. In case $p = 0$, so that there are no operations, $\mathbf{B}$ is
called a <u>relational structure</u>. B is the <u>universe</u> of $\mathbf{B}$, denoted by $|\mathbf{B}|$.

Associated with the similarity type α of $\mathbf{B}$ is the following calculus L_α,
called the <u>monadic second-order language of type</u> α (or, for brevity, simply
"the language of $\mathbf{B}$"). The logical constants of L_α are $\approx$ (equality), the
usual propositional connectives $(\wedge, \vee, \neg, \rightarrow, \leftrightarrow)$, and the quantifiers $\forall$ and $\exists$.
The nonlogical constants of L_α are: For $i < p$, a m_i-ary operation symbol $\underline{O}_i$,
and for $j < q$, a n_j-ary relation symbol $\underline{R}_j$. (For purposes of clarity, when a
structure has an operation O or relation R, we endeavor to use the correspond-
ing boldface letter $\underline{O}$ or $\underline{R}$ as its representative in the formal language. This

is not always desirable, and exceptions to this rule will be made clear when they occur.) There are individual variables x, y, z, ..., and monadic predicate (set) variables X, Y, Z, ... Quantification over either kind of variable is permitted. The notation $y \in X$, read "y is a member of X," will be used instead of the more usual Xy or $X(y)$. The notion of a term, or an atomic formula, and the notion of a variable being free in a f rmula, are understood in the usual way. A sentence is a formula without free variables. If F is a formula of L_α, when we write, e.g., $F(y,x)$, we mean that the variables y, x occur free in F, but we do not exclude the possibility that F has other free variables. If y', x' are any other variables, then when we write, e.g., $F' \equiv F(y',x')$, we mean that F' is obtained from F by substituting y' for each free occurrence of y and x' for each free occurrence of x, while making suitable systematic changes of the bound variables of F so as to avoid "conflicts of variables."

Relative to a given structure $\mathfrak{B}$ of similarity type α and a given interpretation of the individual and set variables, the notions of truth and satisfaction are defined in the usual way. The individual variables will always be interpreted as elements of the universe $|\mathfrak{B}|$. We shall consider two different interpretations of the set variables. In the strong interpretation, set variables range over arbitrary subsets of the universe, while in the weak interpretation, only finite sets are admitted as possible interpretations of the set variables. The strong second-order theory of $\mathfrak{B}$, SS($\mathfrak{B}$), is the set of sentences of L_α which are true under the strong interpretation, and the weak second-order theory of $\mathfrak{B}$, WS($\mathfrak{B}$), is the set of sentences true under the weak interpretation. An elementary formula is a formula without occurrences of set variables, and ET($\mathfrak{B}$) is the set of elementary sentences true in $\mathfrak{B}$.

If C is any class of formulas, we say that a formula F is a Boolean combination of members of C if F is a member of the least class C' containing C and such that whenever G, $H \in C'$, then $G \wedge H$, $G \vee H$, and $\neg G$ are also members of C'.

The symbols Σ, Π are used for iterated disjunction and iterated conjunction, respectively; e.g., for $\ell > 0$, $\Sigma_{i<\ell} F_i$ denotes the formula $F_0 \vee \ldots \vee F_{\ell-1}$.

Let $F(\underline{x},\underline{y},\underline{X})$ be a formula of L_α with exactly the free variables $\underline{x}$, $\underline{y}$, $\underline{X}$, and let x, $y \in |\mathfrak{A}|$ and $X \subseteq |\mathfrak{A}|$. Then $F(x,y,X)$ means that F is satisfied when $\underline{x}$, $\underline{y}$, $\underline{X}$ are interpreted as x, y, X, respectively. Of course, we must also specify whether the weak or strong interpretation is to be used. This will always be clear from context. In fact, we shall rarely use the strong interpretation except in Section 5; thus, in the absence of specific notice to the contrary, the reader may assume that the weak interpretation is intended.

DEFINITION 3.1. Let $0 < p < \omega$. The algebra of p successors is the algebraic structure

$$\mathfrak{N}_p = \langle N_p, S_o, \ldots, S_{p-1} \rangle$$

where $N_p = \{a_o, \ldots, a_{p-1}\}^*$ and for each $i < p$, S_i is the unary operation defined by

$$S_i(x) = xa_i \text{ for all } x \in N_p.$$

The monadic second-order language associated with $\mathfrak{N}_p$, i.e., $L_{\langle\langle 0,\ldots,p-1\rangle\rangle}$, will be denoted by L_p, and its p unary operation symbols by $\underline{S}_o$, $\ldots$, $\underline{S}_{p-1}$.

In the remainder of this section, we shall assume, except where otherwise specified, that p is a fixed but arbitrary positive integer.

There are two main steps in our discussion leading up to Theorem 3.7 and Corollary 3.8: First, we develop a normal form for formulas in L_p, and second, we correlate a recognizable set A with each formula F in normal form, and show that F is satisfiable in $\mathfrak{N}_p$ if and only if $A \neq \emptyset$.

The terms ψ of L_p are all of the form $\psi = \underline{S}_{i_0} \ldots \underline{S}_{i_{k-1}}(\underline{x})$, $k < \omega$, for some individual variable $\underline{x}$; the integer k is the rank of ψ. We say that two formulas F, G of L_p are equivalent and write $F \sim G$ if they have the same free variables and the universal closure of $F \leftrightarrow G$ is in $WS(\mathfrak{N}_p)$.

LEMMA 3.2. Every formula F of L_p is equivalent to a formula G which contains no occurance of the equality symbol, nor of any term of rank > 1.

PROOF. By iterative applications of the two rules

(1) $\psi \approx \varphi \sim \forall \underline{Y}[\psi \in \underline{Y} \leftrightarrow \varphi \in \underline{Y}]$

and

(2) $\underline{X} \approx \underline{Y} \sim \forall \underline{u}[\underline{u} \in \underline{X} \leftrightarrow \underline{u} \in \underline{Y}]$

where ψ, φ are any terms and $\underline{X}$, $\underline{Y}$ are any set variables, we obtain a formula F', with no occurrence of $\approx$, such that $F \sim F'$.

Now suppose that $\underline{S}_{j_0} \ldots \underline{S}_{j_k}(\underline{x})$ is a term of rank > 1 (i.e., k > 0) occurring in F'; this occurrence must be as a part of an atomic formula $\underline{S}_{j_0} \ldots \underline{S}_{j_k}(\underline{x}) \in \underline{Y}$ for some set variable $\underline{Y}$. We note that

(3) $\underline{S}_{j_0} \cdots \underline{S}_{j_k}(\underline{x}) \in \underline{Y} \sim \exists \underline{y}_0 \cdots \exists \underline{y}_k [\forall \underline{z}[\underline{y}_0 \in \underline{z} \leftrightarrow \underline{x} \in \underline{z}]$

$$\wedge \pi_{j=1}^{k} (\forall \underline{z}[\underline{y}_j \in \underline{z} \leftrightarrow \underline{S}_{i_j}(\underline{y}_{j-1}) \in \underline{z}]) \wedge \underline{y}_k \in \underline{Y}]$$

The desired formula G may now be obtained from F' by repeated applications of
rule (3).

DEFINITION 3.3. A <u>principal</u> n-<u>formula</u> is a formula in L_p of the form

$$\exists \underline{x}[\pi_{i<n}(F_i \wedge \pi_{j<p} G_{i,j})],$$

where $\underline{x}$ is any individual variable, and for some n distinct set variables,
$\underline{X}_0, \ldots, \underline{X}_{n-1}$, each F_i is either $\underline{x} \in \underline{X}_i$ or $\neg \underline{x} \in \underline{X}_i$, and each $G_{i,j}$ is either
$\underline{S}_j(\underline{x}) \in \underline{X}_i$ or $\neg \underline{x} \in \underline{X}_i$. A formula

$$(Q \underline{X}_n) \cdots (Q \underline{X}_{n+m-1}) H (\underline{X}_0, \ldots, \underline{X}_{n+m-1})$$

where H is a Boolean combination of principal n + m-formulas and each $(Q \underline{X}_j)$
is, independently of j, either $\forall \underline{X}_j$ or $\exists \underline{X}_j$, in <u>normal form</u>.

LEMMA 3.4. Every formula of L_p with no free individual variables is equivalent
to a formula in normal form.

<u>PROOF</u>. Let $F(\underline{X}_0, \ldots, \underline{X}_{n-1})$ be any formula with exactly the distinct free set
variables $\underline{X}_0, \ldots, \underline{X}_{n-1}$. By 3.2, $F \sim F'$ where F' is a formula with no occurrence
of $\approx$ nor of any term of rank > 1. Note that F' necessarily contains at least
one set variable. The two equivalences,

$$\forall \underline{x} \forall \underline{Y} H \sim \forall \underline{Y} \forall \underline{x} H,$$

$$\exists \underline{x} \forall \underline{Y} H \sim \exists \underline{X} \forall \underline{Y}[\forall \underline{x}[\underline{x} \in \underline{X} \rightarrow H] \wedge \exists \underline{x}[\underline{x} \in \underline{X}]],$$

apply to any formula H in which $\underline{X}$ does not occur free. By iterative applications of these equivalences we obtain a formula $F'' \sim F'$ such that no set variable quantifier in F'' occurs within the scope of any individual variable quantifier. Now F'' is equivalent to its prenex normal form, i.e.,

$$F'' \sim (Q\ \underline{X})_n \cdots (Q\ \underline{X})_{n+m-1}\ F'''(\underline{X}_o, \ldots, \underline{X}_{n+m-1})$$

where $\underline{X}_n, \ldots, \underline{X}_{n+m-1}$ are all distinct, each $(Q\ \underline{X}_j)$, $j = n, \ldots, n + m - 1$, is either $\forall\ \underline{X}_j$ or $\exists\ \underline{X}_j$, and $F'''(\underline{X}_o, \ldots, \underline{X}_{n+m-1})$ contains no set variable quantifier. To complete the proof, we must show that F''' is equivalent to a Boolean combination of principal $n + m$-formulas.

Let C be the class of Boolean combinations of principal $n + m$-formulas and atomic formulas $\underline{y} \in \underline{X}_j$ or $\underline{S}_i(\underline{y}) \in \underline{X}_j$, where $j < m$, $i < p$, and $\underline{y}$ is any individual variable. Let C' be the class of formulas equivalent to formulas in C. That $F''' \in C'$ is shown by induction; we will only discuss the existential quantifier step, namely, we assume that $G \in C$ and show that $\exists\ \underline{x}\ G \in C'$. Of course, if $\underline{x}$ does not occur in G then $\exists\ \underline{x}\ G \sim G$. Otherwise, G may be put in its disjunctive normal form, $\Sigma_{\ell < k} G_\ell$, and the quantifier distributed:

$$\exists\ \underline{x}\ G \sim \Sigma_{\ell < k}\ \exists\ \underline{x}\ G_\ell,$$

where each G_ℓ is a conjunction in C. For each $\ell < k$, all the conjuncts of G_ℓ in which $\underline{x}$ does not occur free may be passed outside the scope of the quantifier; i.e., we apply the rule that for any formula H and any $\underline{y} \neq \underline{x}$,

$$\exists\ \underline{x}\ [H \wedge \underline{y} \in \underline{X}] \sim \exists\ \underline{x}\ H \wedge \underline{y} \in \underline{X},$$

or the similar rules concerning conjuncts $\neg\ \underline{y} \in \underline{X}$, $\underline{S}_1(\underline{y}) \in \underline{X}$, or $\neg\ \underline{S}_1(\underline{y}) \in \underline{X}$. Finally, if for any $j < n + m$ such that neither $\underline{x} \in \underline{X}_j$ nor $\neg\ \underline{x} \in \underline{X}_j$ occurs as a conjunct within the scope of $\exists\ \underline{x}$, then $\underline{x} \in \underline{X}_j \vee \neg\ \underline{x} \in \underline{X}_j$ may be inserted as a conjunct and the distributive laws again applied; a similar treatment applies when neither $\underline{S}_1(\underline{x}) \in \underline{X}_j$ nor $\neg\ \underline{S}_1(\underline{x}) \in \underline{X}_j$ occurs within the scope of $\exists\ \underline{x}$. The resulting formula, G'_ℓ, is in C, $G'_\ell \sim \exists\ \underline{x}\ G_\ell$, and the variable $\underline{x}$ occurs in G'_ℓ only as the bound variable in principal $n + m$-formulas. Since $\exists\ \underline{x}\ G$ is equivalent to a Boolean combination of such formulas G'_ℓ, we have $\exists\ \underline{x}\ G \in C'$.

Lemma 3.4 generalizes Elgot's Lemma 1, Section 5.5, in [10] to L_p for $p > 1$. The proof uses essentially the same ideas. We may note in passing that the proof of 3.4 makes no use of special properties of the operations or $\mathfrak{N}_p$. The lemma can be proved for any monadic second-order formal language in which there are no nonlogical constants with more than one argument place for individual variables: We can even introduce higher-type predicte constants with one individual variable argument place and one predicate variable argument place; the treatment of such higher-type constants would be formally similar to the treatment of "$\in$".

For each n, let Σ_n be the set of n-termed sequences with terms in the set $\{0,1\}$; $0^{(n)}$ denotes the n-termed sequence consisting entirely of 0's.

The order ot trees and acceptors discussed in this section is assumed to be p; thus, if Π is an alphabet, $\Pi^\#$ is the set of Π trees of order p.

DEFINITION 3.5. Let $\tau \in \Sigma_n^\#$, and let $X_0,\ \ldots,\ X_{n-1}$ be finite subsets of N_p. Then τ represents $X_0,\ \ldots,\ X_{n-1}$ if, for each $i < n$ and any $w \in N_p$, $w \in X_i$ if and only if the i-th term of τ_w is 1.

Every tree $\tau \in \Sigma_n^{\#}$ represents exactly one sequence of sets $X_o, \ldots, X_{n-1} \subseteq N_p$, and every sequence of finite sets $X_o, \ldots, X_{n-1} \subseteq N_p$ is represented by some tree $\tau \in \Sigma_n^{\#}$. The tree τ is not uniquely determined by the sets $X_o, \ldots, X_{n-1}$, however; e.g., if $\tau_w = 0^{(n)}$ for some $w \in fr(\tau)$, then both τ and $\tau[w/\Lambda]$ represent the same sets. Nevertheless, there is always just one minimal Σ_n-tree which represents the given sets $X_o, \ldots, X_{n-1}$; it may be obtained as follows: Let τ be any tree representing the sets $X_o, \ldots, X_{n-1}$, and put

$$\tau_w' = \tau_w \text{ if } \tau \backslash w \notin \{0^{(n)}\}^{\#},$$

$$= \epsilon \text{ otherwise;}$$

then τ' is the minimal tree representing $X_o, \ldots, X_{n-1}$.

Let us say that two Σ_n-trees are equivalent if they represent the same sequence of sets. If A is any set of Σ_n-trees, we denote by $cl(A)$ the set of all trees equivalent to some tree in A, and by $mnl(A)$ the set of minimal trees in $cl(A)$.

LEMMA 3.6. If $A \subseteq \Sigma_n$ is recognizable, then so are $cl(A)$ and $mnl(A)$.

PROOF. Let $\mathfrak{A} = \langle S, t, \bar{s}, D \rangle$ be a Σ_n-tree acceptor such that $T(\mathfrak{A}) = A$. First, suppose that $A = mnl(A)$. Let s^* be a new state not in S, and put $\mathfrak{B} = \langle S \cup \{s^*\}, t', s^*, D' \rangle$, where $D' = D \cup \{s^*\}$ if $\bar{s} \in D$, $D' = D$ otherwise, and t' is defined as follows:

$$t'(s^*, \ldots, s^*, 0^{(n)}) = s^*,$$

and if either $\sigma \neq 0^{(n)}$ or some s_i is not s^*, then

$$t'(s_0, \ldots, s_{p-1}, \sigma) = t(s'_0, \ldots, s'_{p-1}, \sigma)$$

where $s'_i = s_i$ if $s_i \neq s^*$ and $s'_i = \bar{s}$ if $s_i = s^*$. With $A = mnl(A)$, it is easily seen that $T(\mathfrak{B}) = cl(A)$.

Now consider the case that $A \neq mnl(A)$. Since obviously $cl(A) = cl(mnl(A))$, it is, in view of what has already been proved, sufficient to show that $mnl(A)$ is recognizable. Again, assume $A = T(\mathfrak{U})$, $\mathfrak{U} = \langle S, t, \bar{s}, D \rangle$. Let $\mathfrak{C} = \langle S, t'', I, D \rangle$, where for all $s_0, \ldots, s_{p-1} \in S$, $\sigma \in \Sigma_n$,

$$t''(s_0, \ldots, s_{p-1}, \sigma) = \{t(s_0, \ldots, s_{p-1}, \sigma)\}$$

and

$$I = \{s : \bar{t}(\tau) = s \text{ for some } \tau \in \{0^{(n)}\}^{\#}\}.$$

Then $\mathfrak{C}$ is a non-deterministic Σ_n-tree acceptor with the following two properties: (i) every $\tau \in T(\mathfrak{C})$ is equivalent to some member of A, and (ii) $mnl(A) \subseteq T(\mathfrak{C})$. Thus, $mnl(A)$ consists of those $\tau \in T(\mathfrak{C})$ such that $\tau_w \neq 0^{(n)}$ for each $w \in fr(\tau)$. It follows from 1.14 and 1.15 that $mnl(A)$ is recognizable.

Note that in the proof of 3.6, the construction of the nondeterministic tree acceptor $\mathfrak{C}$ is effective, since in the definition of I we may restrict consideration to those trees $\tau \in \{0^{(n)}\}$ which are of depth $< \bar{\bar{s}}$.

If $F(\underline{X}_0, \ldots, \underline{X}_{n-1})$ is a formula of L_p with exactly the free variables $\underline{X}_0, \ldots, \underline{X}_{n-1}$, then we denote by $T(F)$ the set of those minimal Σ_n-trees which represent a sequence $X_0, \ldots, X_{n-1}$ such that $F(X_0, \ldots, X_{n-1})$.

THEOREM 3.7. $T(F)$ is recognizable for every formula F with exactly the free variables $\underline{X}_0, \ldots, \underline{X}_{n-1}$.

PROOF. It is sufficient, by 3.4, to assume that F is in normal form. Our proof is by induction on the length of F. In each case of the induction we shall exhibit a recognizable set equal to $T(F)$; the reader should encounter no difficulty in supplying the simple argument which establishes this equality.

If F is a principal n-formula, say $\exists \, \underline{x}(\pi_{i<n}(F_i \wedge \pi_{j<p} G_{ij}))$, where F_i, G_{ij} are as in Definition 3.3, let $\sigma \in \Sigma_n$ be defined by the condition

the i-th term of σ is 1 iff F_i is $\underline{x} \in \underline{X}_i$,

and for $j = 0, \ldots, p - 1$ let $\sigma_j \in \Sigma_n$ be defined by the condition

the i-th term of σ_j is 1 iff G_{ij} is $\underline{S}_j(\underline{x}) \in \underline{X}_i$.

Now if $\sigma = \sigma_0 = \cdots = \sigma_{p-1} = 0^{(n)}$, then $T(F) = \{\Lambda\}$. But if at least one of $\sigma, \sigma_0, \ldots, \sigma_{p-1}$ is not $0^{(n)}$, then we have

$$T(F) = mnl(E_{\Sigma_n}(\sigma, \sigma_0, \ldots, \sigma_{p-1})),$$

which is recognizable by 1.15 and 3.6.

If F is a Boolean combination of principal n-formulas, then we need merely note that the recognizable sets are closed under $\cap$, $\cup$, and $\sim$, e.g., if F is $G \vee H$ and $T(G)$, $T(H)$ are recognizable, then so is $T(F) = T(G) \cup T(H)$.

Finally, suppose that F is $\exists \, X_{-n+1} \, G(\underline{X}_{-o}, \ldots, \underline{X}_{-n})$ and $T(G)$ is a recognizable subset of $\Sigma_{n+1}^{\#}$. Let g be the projection of $\Sigma_{n+1}^{\#}$ into $\Sigma_n^{\#}$ defined by

$$g(\langle k_o, \ldots, k_n \rangle) = \langle k_o, \ldots, k_{n-1} \rangle \text{ for each } \langle k_o, \ldots, k_n \rangle \in \Sigma_{n+1}.$$

Then $T(F) = \text{mnl}(g(T(G)))$, and this is recognizable by 1.13, 1.15, and 3.6.

The characterization of the recognizable sets developed in 1.13, 1.14, and 1.15 is not essential to the proof of 3.7--one can also give direct constructions of tree acceptors $\mathfrak{U}$ such that $T(\mathfrak{U}) = T(F)$ for each of the various forms of the formula F.

<u>COROLLARY</u> 3.8. $WS(\mathfrak{R}_p)$ is decidable for every finite p.

<u>PROOF.</u> If F is an arbitrary sentence of L_p, then, by 3.4, F is equivalent to a sentence F' which is in normal form. F' has at least one set variable; suppose, for example, that F' is $\exists \underline{X}\ G(\underline{X})$. (In case F' is $\vee \underline{X}\ G(\underline{X})$, we consider instead $\exists \underline{X} \neg G(\underline{X}) \sim \neg F'$.) Now F', and hence F, is a member of $WS(\mathfrak{R}_p)$ iff $T(G)$ is not empty. But $T(G) = \emptyset$ is effectively decidable by Corollary 1.12. We need only verify that F' and $T(G)$ can be effectively obtained. This is accomplished by examination of the proofs of 3.4 and the results in Section 1.

We shall devote the remainder of this section to a discussion of applications of 3.7, deferring consideration of the many applications of 3.8 until Sections 4 and 5.

Theorem 3.7 has a converse: Roughly speaking, "every recognizable set can be expressed in the form $T(F)$ for some formula F." This statement fails to be strictly true only because the underlying alphabets of the sets $T(F)$ are not arbitrary, but are always one of the Σ_n. In the following theorem, we restrict consideration to the alphabets $\Sigma_n \sim \{0^{(n)}\}$, denoted by Δ_n, in order to avoid

difficulties associated with th. ambiguities in the representation of sequences of sets by Σ_n-trees. The Δ_n still provide alphabets of arbitrarily large finite cardinality.

THEOREM 3.9. Every set recognizable over some alphabet Δ_n, $n > 0$, can be expressed in the form $T(F)$ for some formula F.

PROOF. For each $\sigma \in \Sigma_n$, let F^σ be $\pi_{i<n} F_i$, where F_i is $\underline{x} \in \underline{X}_i$ if the i-th term of σ is 1, and $F_i \neg \underline{x} \in \underline{X}_i$ otherwise. Similarly, let G^σ_j be $\pi_{i<n} G_{ij}$ where G_{ij} is $\underline{S}_j(\underline{x}) \in \underline{X}_i$ if the i-th term of σ is 1 and G_{ij} is $\neg \underline{S}_j(\underline{x}) \in \underline{X}_i$ otherwise. Put

$$H_n \equiv \neg \ \exists \ \underline{x}[F^{0^{(n)}} \wedge \neg \ \pi_{j<p} \ G_j^{0^{(n)}}].$$

Then $H_n(X_0, \ldots, X_{n-1})$ holds iff the minimal tree representing $X_0, \ldots, X_{n-1}$ is a tree over Δ_n.

Now let A be any recognizable subset of $\Delta_n^\#$. If $A = \Delta_n^\#$, then $A = T(H_n)$. If $A = E_{\Delta_n}(\sigma, \sigma_0, \ldots, \sigma_{p-1})$ for some $\sigma, \sigma_0, \ldots, \sigma_{p-1} \in \Delta_n$, we put

$$F \equiv H_n \wedge \exists \ \underline{x}[F^\sigma \wedge \pi_{j<p} \ G_j^{\sigma_j}]$$

and then we have $A = T(F)$. In case one or more of the σ_j is ϵ, we need merely replace $G_j^{\sigma_j}$ by $G_n^{0^{(n)}}$.

Now suppose that $A = B \cap C$ for some recognizable sets B, C. In general, the alphabets for B and C may properly include Δ_n. However, it is easily seen that we can find $m \geq n$ and recognizable sets B', C' over Δ_m such that $P(A) = B' \cap C'$ where p is the projection of $\Delta_n^\#$ into $\Delta_m^\#$ defined by

$$P(\langle r_o, \ldots, r_{n-1} \rangle) = \langle r_o, \ldots, r_{n-1}, 0, \ldots, 0 \rangle$$

for each $\langle r_o, \ldots, r_{n-1} \rangle \in \Delta_n$. There exist formulas F', F'' such that $T(F') = B'$ and $T(F'') = C'$. We let

$$F = \exists \underline{X}_n \ldots \exists \underline{X}_{m-1} [\Pi_{n \leq i < m} \neg \exists \underline{x} [\underline{x} \in \underline{X}_i] \wedge F' \wedge F''],$$

and obtain $A = T(F)$. The cases $A = B \cup C$ and $A = B \sim C$ are handled in like manner.

Finally suppose that $A = p(B)$ for some projection p and recognizable set B. Without loss of generality, we may assume that the underlying alphabet of B is some Δ_m, $m \geq n$. There exists a formula $G = G(\underline{X}_o, \ldots, \underline{X}_{m-1})$ such that $T(G) = B$. Let $\underline{Y}_o$, ..., $\underline{Y}_{m-1}$ be distinct new variables which do not occur in G, and put

$$F = \exists \underline{Y}_o \ldots \exists \underline{Y}_{m-1} [G(\underline{Y}_o, \ldots, \underline{Y}_{m-1}) \wedge \vee \underline{x} [\Pi_{\sigma < \Delta_m} (F^\sigma(\underline{Y}_o, \ldots, \underline{Y}_{m-1})$$
$$= F^{p(\sigma)}(\underline{X}_o, \ldots, \underline{X}_{n-1})) \wedge$$
$$(F^{o(n)}(\underline{Y}_o, \ldots, \underline{Y}_{m-1}) = F^{)(n)}(\underline{X}_o, \ldots, \underline{X}_{n-1}))]].$$

Then $T(F) = A$.

Theorem 3.9 now follows from 1.15.

COROLLARY 3.10. A subset L of Δ_n^* is a context-free language if and only if $L = Q(T(F(X_o, \ldots, X_{n-1})))$ for some formula F.

A subset $X \subseteq N_p$ is weak second-order definable in $\mathfrak{R}_p$ if for some formula $F(\underline{x})$, with exactly the one free individual variable $\underline{x}$, X is the set of $x \in N_p$ such that $F(x)$ holds under the weak interpretation. We say then that X is defined by F in $\mathfrak{R}_p$; similarly, we speak of subsets of $N_p \times N_p$ as defined by formulas $F(\underline{x}, \underline{y})$, and so on.

COROLLARY 3.11. A subset of N_p is weak second-order definiable in $\mathcal{R}_p$ if and only if it is regular.

PROOF. In view of 3.7, 3.9, 1.16, and the remarks at the end of Section 1, we need only show that, for any formula F with exactly the free variables $X_0, \ldots, X_{n-1}$, $\bigcup_{\tau \in T(F)} fr(\tau)$ is definable in $\mathcal{R}_p$. In fact, if

$$G(x, X_0, \ldots, X_{n-1}) = \Sigma_{i<n}[x \in X_i] \wedge \Pi_{i<n,\, j<p}[\neg \underline{s}_j(x) \in X_i],$$

then $\bigcup_{\tau \in T(F)} fr(\tau)$ is defined by

$$\exists X_0 \ldots \exists X_{n-1}[F(X_0, \ldots, X_{n-1}) \wedge G(x, X_0, \ldots, X_{n-1})].$$

In this paper we have identified three distinct methods of defining a recognizable set $A \subset \Delta_n^{\#}$, $n > 0$: A may be expressed in any of the forms

(I)   The result of a finite sequence of applications of projections and Boolean set operations starting with sets of the form $E_i(\sigma, \sigma_0, \ldots, \sigma_{p-1})$;

(II) $T(\mathfrak{A})$ for some tree acceptor $\mathfrak{A}$;

(III) $T(F)$ for some formula $F(X_0, \ldots, X_{n-1})$ of L_p.

These may be compared with the following methods of defining a regular set $B \subset (a_0, \ldots, a_{p-1})^*$: B may be expressed in any of the forms

(I')   The result of a finite sequence of applications of the operations $\cdot$, $*$, $\cup$, $\cap$, and $\sim$ starting with finite sets of words;

(II') $T(\mathfrak{A})$ for some finite automaton $\mathfrak{A}$;

(III') $\{x : F(x)\}$ for some formula $F(x)$ of L_p.

The equivalence of the forms of definition (I), (II), and (III) has been established in this paper by 1.14, 1.15, 3.7, and 3.9, while the equivalence of (I'), (II'), and (III') is a consequence of well-known results in the literature and Corollary 3.11. Examination of the proofs of these equivalences discloses that each of them is completely effective.

LEMMA 3.12. (i) If a definition of a recognizable set $A \subseteq \Delta_n^*$, $n > 0$, is given in one of the forms (I), (II), (III), then definitions of A in each of the other two forms can be effectively obtained.

(ii) If a definition of a regular set $B \subseteq N_p$, $p > 0$, is given in one of the forms (I'), (II'), (III'), then definitions in each of the other two forms can be effectively obtained.

In [1] Buchi considered the "very weak second-order theory" of $\mathfrak{N}_p$, in which the set variables range, not over arbitrary finite subsets of N_p, but only over those finite subsets which are chains with respect to the initial segment relation. Theorem 10 of his paper states that the class of subsets of N_p definable in the very weak second-order theory coincides with the regular subsets of N_p. Thus we see that, from the point of view of defining subsets of N_p, the weak second-order theory is no more powerful than the very weak second-order theory.

SECTION 4. APPLICATIONS

In this section we shall apply the results of Section 3 to establish the decidability of a variety of weak second-order theories. The same general method will be used in nearly all cases: the decidability of $WS(\mathfrak{N})$ is proved by interpreting $WS(\mathfrak{N})$ into $WS(\mathfrak{R}_p)$ for some p (usually $p = 2$). This interpretation is based upon a definition in $WS(\mathfrak{R}_p)$ of a substructure of $\mathfrak{R}_p$ isomorphic to the given structure $\mathfrak{N}$. For example, if $\mathfrak{N} = \langle A, O, R \rangle$ where O is a binary operation and R is a binary relation, and there are formulas $F(\underline{x}), G(\underline{x}, \underline{y}, \underline{z}), H(\underline{x}, \underline{y})$ such that

$$\mathfrak{N} \cong \langle A', O', R' \rangle$$

where

$$x \in A \text{ iff } F(x) \text{ holds in } \mathfrak{R}_p,$$

$$O'(x,y) = z \text{ iff } G(x,y,z) \text{ holds in } \mathfrak{R}_p,$$

$$R'(x,y) \text{ iff } H(x,y) \text{ holds in } \mathfrak{R}_p$$

then we say that the triple (F, G, H) is a <u>weak second-order definition</u> of $\mathfrak{N}$ in $\mathfrak{R}_p$. It follows from the existence of such a definition that $WS(\mathfrak{N})$ is interpretable in $WS(\mathfrak{R}_p)$, and hence that $WS(\mathfrak{N})$ is decidable. (For further information on interpretation of theories, the reader may consult [25].)

<u>THEOREM 4.1</u>. For each finite $p > 0$, $\mathfrak{R}_p$ is weak second-order definable in $\mathfrak{R}_2$.

<u>PROOF</u>. Let $A = (\{a\} \cdot (\bigcup_{i<p} \{b\}^i))^*$. Then A is a regular set, and, by 3.12, a formula $F(\underline{x})$ defining A in $\mathfrak{R}_2$ can be effectively obtained. Let the terms

ψ_n, $n < \omega$, be defined by recursion: $\psi_0(\underline{x}) = \underline{x}$, $\psi_{n+1}(\underline{x}) = \underline{S}_1(\psi_n(\underline{x}))$ for each n. Putting $G_j(\underline{x},\underline{y}) = \underline{y} \approx \underline{S}_0(\psi_j(\underline{x}))$, we have that $\langle F, G_0, \ldots, G_{p-1} \rangle$ is a definition of $\mathfrak{R}_p$ in $\mathfrak{R}_2$.

Let $N_\omega = \{a_0, a_1, \ldots\}^*$, for $x \in N_\omega$ and $n < \omega$, let $S_n(x) = xa_n$, and let $\mathfrak{R}_\omega = \langle N_\omega, S_0, \ldots \rangle$. There is a formula $F_\omega(\underline{x})$ defining the regular set $(\{a\} \cdot \{b\}^*)^*$, and if we let $G_j(\underline{x},\underline{y})$ be as in the proof of 4.1, we find that $\langle F_\omega, G_0, G_1, \ldots \rangle$ is a weak second-order definition of $\mathfrak{R}_\omega$ in $\mathfrak{R}_2$, and hence,

$$\text{WS}(\mathfrak{R}_\omega) \text{ is decidable.}$$

$\text{WS}(\mathfrak{R}_\omega)$ is not as rich a theory as one might wish; for example, even the simple relation "$x = S_n(y)$ for some n" is not definable in it. We can, however, add a further relation to $\mathfrak{R}_\omega$ and obtain more satisfactory results Let $\text{Is}(x,y)$ hold iff x is an initial segment of y, and put $\mathfrak{R}'_\omega = \langle N_\omega, \text{Is}, S_0, S_1, \ldots \rangle$.

THEOREM 4.2 (i) $\mathfrak{R}'_\omega$ is weak second-order definable in $\mathfrak{R}_2$ and hence $\text{WS}(\mathfrak{R}'_\omega)$ is decidable.

(ii) For each $p > 0$, $\mathfrak{R}_p$ is weak second-order definable in $\mathfrak{R}'_\omega$.

(iii) A set $X \subseteq \{a_0, \ldots, a_{p-1}\}^*$ is regular if and only if it is weak second-order definable in $\mathfrak{R}'_\omega$.

PROOF. A definition of $\mathfrak{R}'_\omega$ in $\mathfrak{R}_2$ is $\langle F_\omega, I, G_0, G_1, \ldots \rangle$ where F_ω, G_0, G_1, ... are as above and $I(\underline{x},\underline{y})$ is

$$\vee \underline{X}[\underline{y} \in \underline{X} \wedge \vee \underline{z}[\underline{S}_0(\underline{z}) \in \underline{X} \vee \underline{S}_1(\underline{z}) \in \underline{X} \rightarrow \underline{z} \in \underline{X}] \rightarrow \underline{x} \in \underline{X}] \wedge F_\omega(\underline{x}) \wedge F_\omega(\underline{y}).$$

To prove part (ii), it is sufficient to note that, for each $p > 0$, the formula

$$\forall \underline{y}[I(\underline{y},\underline{x}) \to \underline{y} \approx \underline{x} \vee \Sigma_{j<p} I(\underline{s}_j(\underline{y}),\underline{x})]$$

defines the set $\{a_o,\ldots,a_{p-1}\}^*$ om $\mathcal{R}_\omega$.

Let the mapping $f : \{a_o,a_1,\ldots\}^* \to \{a,b\}^*$ be defined by $f(a_j) = ab^j$, $j = 0, 1, \ldots$ and $f(uv) = f(u)f(v)$ for all words u, v. (f is simply the isomorphism which establishes that $\mathcal{R}'_\omega$ is defined in $\mathcal{R}_2$ by $\langle F_\omega, I, G_o, \ldots\rangle$.) Assume $p > 0$, and let f_p be the restriction of f to $\{a_o,\ldots,a_{p-1}\}^*$. A generalized sequential machine (as defined in Ginsburg and Rose [16]), which effects the mapping f_p can easily be constructed, so that, by a theorem in [16], a set $X \subset \{a_o,\ldots,a_{p-1}\}^*$ is regular if and only if $f_p(X)$ is regular. Now we note the following two properties of the definitions of $\mathcal{R}'_\omega$ in $\mathcal{R}_2$ and of $\mathcal{R}_p$ in $\mathcal{R}'_\omega$:

(1)   if X is definable in $\mathcal{R}'_\omega$, then $f(X)$ is definable in $\mathcal{R}_2$;

(2)   if X is definable in $\mathcal{R}_p$, then X is definable in $\mathcal{R}'_\omega$.

Suppose that $X \subset \{a_o,\ldots,a_{p-1}\}^*$. If X is regular, then X is definable in $\mathcal{R}_p$, so by (2), X is definable in $\mathcal{R}'_\omega$. Conversely, if X is definable in $\mathcal{R}'_\omega$, then, by (1), $f(X) = f_p(X)$ is definable in $\mathcal{R}_2$. But this implies that $f_p(X)$ is regular, whence X is regular also.

Theorem 4.2 (i), in a somewhat different form, was obtained by J.W. Thatcher [26]. Theorem 4.2 (iii) improves 3.11 by giving a single decidable theory, $WS(\mathcal{R}'_\omega)$, within which every regular set may be defined (subject to the restriction that the underlying alphabet be a subset of $\{a_o,a_1,\ldots\}$).

Let α be any order type. By $WS(\alpha)$ we mean $WS(\mathfrak{A})$ where $\mathfrak{A} = \langle A, R \rangle$ is any relational structure such that R is an order relation on A of type α. The notion of (weak second-order) definability is extended in the natural way, i.e., a type α is definable in $\mathfrak{N}_2$ if some structure $\mathfrak{A}$ of type α is definable in $\mathfrak{N}_2$. All the definitions of order types in $\mathfrak{N}_2$ we give will be with the aid of the following ordering of N_2:

DEFINITION 4.3. The <u>left-to-right ordering</u> of N_2 is the relation $<$, defined in the weak second-order theory by the formula

$$\underline{x} < \underline{y} \equiv I(\underline{S}_0(\underline{y}),\underline{x}) \vee I(\underline{S}_1(\underline{x}),\underline{y}) \vee \exists \underline{z}[I(\underline{S}_0(\underline{z}),\underline{x}) \wedge I(\underline{S}_1(\underline{z}),\underline{y})].$$

To understand the nature of the "left-to-right ordering," it may be helpful to draw a graphic representation of N_2, similar to Figure 1. For $x, y \in N_2$, we have $x < y$ iff the branch to x procedes leftward from some point on the branch to y (possibly y itself), or, equivalently, the branch to y procedes rightward from some point on the branch to x.

For any class S of order types let $C(S)$ be the closure of S under the order-type operations $+$ (addition), $\cdot$ (multiplication), and $\check{\ }$ (converse). η is the type of the rationals.

THEOREM 4.4. If $S = \{\omega, \eta\} \cup \{0, 1, \ldots\}$ and $\alpha \in C(S)$, then α is weak second-order definable in $\mathfrak{N}_2$, and hence, $WS(\alpha)$ is decidable.

PROOF. We shall show how to obtain, for each $\alpha \in C(S)$, a regular set $A_\alpha \subseteq N_2$ which is ordered of type α by the left-to-right ordering, $<$, and which satisfies the additional condition

(1) if $x, y, \in A_\alpha$, then x is not an initial segment of y.

For each $n < \omega$, let $A_n = \{b^0 a, \ldots, b^{n-1} a\}$, let

$$A_\omega = \{b\}^* \cdot \{a\} \text{ and } A_\eta = \{ab, abb\}^* \cdot \{a\}.$$

If A_α, A_β have been obtained, then $A_{\alpha+\beta} = (\{a\} \cdot A_\alpha) \cup (\{b\} \cdot A_\beta)$ and $A_{\alpha \cdot \beta} = A_\alpha \cdot A_\beta$. $A_{\check{\alpha}} = f(A_\alpha)$, where f is the projection of N_2 onto N_2 such that $f(a) = b$ and $f(b) = a$. It is easily verified that if A_α, A_β satisfy (1), then $A_{\alpha+\beta}$, $A_{\alpha \cdot \beta}$, and $A_{\check{\alpha}}$ also satisfy (1). The proof is completed by induction, showing that each A_α, $\alpha \in C(S)$, is indeed ordered of type α by $<$ (the condition (1) is needed in the case $A_{\omega \cdot \beta}$).

The improvement made by Theorem 4.4 over results known prior to the publication of [7] is simply the inclusion of η in the set S.

Let us say of two order types α, β that $\alpha \equiv_n \beta$ if $WS(\alpha)$, $WS(\beta)$ contain the same sentences with n or fewer qualifiers. Thus, $WS(\alpha) = WS(\beta)$ iff $\alpha \equiv_n \beta$ for every n. In [9], Ehrenfeucht gave a condition[2]—we denote it by ℓ_n—

_____

[2] The condition $\ell_n(\alpha, \beta)$ is defined as follows: We imagine a game between two players, I and II. In the first move, player I selects one of the order types α, β and chooses a finite sequence of types which are initial segments of this one, and player II responds with an equally long sequence of initial segments of the other of α, β; e.g., I chooses $\beta_0, \ldots, \beta_k < \beta$, and II responds with $\alpha_0, \ldots, \alpha_k < \alpha$. In succeeding moves, the two players repeat this process, extending the sequences already obtained. Player II wins if, after n moves, the resulting sequences are order isomorphic; otherwise player I wins. The condition $\ell_n(\alpha, \beta)$ holds just in case player II has a winning strategy

which, when modified to apply to order types instead of structures, yields the
following:

For any two ordinals α, β, if $\ell_n(\alpha, \beta)$, then $\alpha \equiv_n \beta$.

Ehrenfeucht also showed that if α is any ordinal then, for each n, there exists
$\alpha' < \omega^n$ such that $\ell_n(\alpha, \alpha')$. It is not difficult to show that the operations
$+$, $\cdot$, and $\check{}$ preserve the condition ℓ_n; namely, for any order types α, α',
β, β', if $\ell_n(\alpha, \beta)$ and $\ell_n(\alpha', \beta')$ then $\ell_n(\alpha+\alpha', \beta+\beta')$, $\ell_n(\alpha \cdot \alpha', \beta \cdot \beta')$, and
$\ell_n(\check{\alpha}, \check{\beta})$. In this way, we obtain

COROLLARY 4.5. Let OR denote the class of all ordinals. If $\alpha \in C(OR \cup \{\eta\})$,
then $WS(\alpha)$ is decidable.

Corollary 4.5 improves a result in the literature (see [3], [11], and
[9]) by which $WS(\alpha)$ is decidable for every ordinal α. In [12] it is stated
that Ehrenfeucht had obtained a decision method for the theory of ordinal
addition; from this result, the decidability of $WS(\alpha)$ for every ordinal α
follows at once by Theorem 10.1 of [12]. Ehrenfeucht never published his proof,
however; and later, a proof of these results was published by Buchi [3].

As this paper was being written, the author learned (by personal communica-
tion) that M.O. Rabin had found a proof of the decidability of $SS(\mathfrak{R})$. (This
proof has since been published in [22].) It is worth noting that all
our theorems concerning definability of order types continue to hold in

the context of the strong second-order theory. (To show this, one has only to exhibit a formula $F(\underline{X})$ such that for $X \subseteq N_2$, $F(X)$ holds in the strong interpretation iff X is finite.) Because every denumerable order type can be embedded in the rationals, Rabin's result at once yields, as he has pointed out, the decidability of the strong monadic second-order theory of countable linear orderings, thus considerably improving a result of Buchi [4], to the effect that the corresponding theorem holds for countable well-orderings.

SECTION 5. DECISION PROBLEMS OF LOCALLY FREE ALGEBRAS

In this section we shall apply Corollary 3.8 to prove the decidability of
the weak second-order theory of a general class of structures which includes
the $\mathfrak{R}_p$ as particular cases. We make essential use of Buchi's theorem on the
decidability of $SS(\mathfrak{R}_1)$ [2], and of the generalized products of Feferman and
Vaught [12].

Let $\mathfrak{A} = \langle A, O_0, \ldots, O_{p-1}, R_o, \ldots, R_{q-1} \rangle$ be an algebraic structure. If
$\emptyset \neq A' \subseteq A$, then $\mathfrak{E}_{\mathfrak{A}}(A')$ denotes the __subalgebra__ of $\mathfrak{A}$ __generated__ by A': Namely,
$\mathfrak{E}_{\mathfrak{A}}(A') = \langle A'', O_0', \ldots, O_{p-1}', R_0', \ldots, R_{q-1}' \rangle$ where A'' is the least set containing A'
and closed under the operations $O_0, \ldots, O_{p-1}$, and each of O_j', R_j' are the restric-
tions of O_i, R_j, respectively, to the set A''. In case A' consists of a single
element, i.e., $A' = \{x\}$, we write $\mathfrak{E}_{\mathfrak{A}}(x) = \mathfrak{E}_{\mathfrak{A}}(A')$.

If $\mathcal{Q}$ is any class of algebraic structures of a given similarity type, then
$WS(\mathcal{Q})$ is the set of weak second-order sentences true in every element of $\mathcal{Q}$. A
sentence is true in $\mathcal{Q}$ if it is true in every member of $\mathcal{Q}$. If T is any set of
sentences in a language L, then $WS(T) = WS(\mathcal{Q})$, where $\mathcal{Q}$ is the class of structures
$\mathfrak{A}$ having the same type as L and such that $T \subseteq WS(\mathfrak{A})$. Interpretations similar to
these apply to the notiations $SS(\mathcal{Q})$, $ET(T)$, etc.

Let I be any nonempty set and let $\mathfrak{A}_i = \langle A^{(i)}, O_o^{(i)}, \ldots, O_{p-1}^{(i)}, R_o^{(i)}, \ldots, R_{q-1}^{(i)} \rangle$,
$i \in I$, be algebraic structures of the same similarity type, indexed by elements
of the set I, such that the $A^{(i)}$ are pairwise disjoint. The __cardinal sum__ of
the $\mathfrak{A}_i$ is the structure $\mathfrak{A} = \langle A, O_o, \ldots, O_{p-1}, R_o, \ldots, R_{q-1} \rangle$ where $A = \bigcup_{i \in I} A^{(i)}$,
$O_j = \bigcup_{i \in I} O_j^{(i)}$, $j = 0, \ldots, p-1$, and $R_j = \bigcup_{i \in I} R_j^{(i)}$, $j = 0, \ldots, q-1$. In case

the universes of the $\mathfrak{A}_i$ are not disjoint, then we understand their cardinal sum to be the cardinal sum of a set of structures $\mathfrak{A}_i'$, $i \in I$, with $\mathfrak{A}_i' \cong \mathfrak{A}_i$ for each i, which do have mutually disjoint universes.

Let p > 0 and let L be the language with only the operation symbols $\underline{O}_0, \ldots, \underline{O}_{p-1}$, where $\underline{O}_i$ is n_i-ary, $n_i > 0$, for i = 0, ..., p-1. In [17], Mal'cev considered the elementary theory based upon the axioms

$$\underline{O}_i(\underline{x}_1, \ldots, \underline{x}_{n_i}) \not\sim \underline{O}_j(\underline{y}_1, \ldots, \underline{y}_{n_j}), \quad 0 \le i < j < p,$$

$$(\text{I}) \qquad \underline{O}_i(\underline{x}_1, \ldots, \underline{x}_{n_i}) \sim \underline{O}_i(\underline{y}_1, \ldots, \underline{y}_{n_i}) \to \pi_{j=1}^{n_i}(\underline{x}_j \sim \underline{y}_j), \quad 0 \le i < p,$$

$\underline{x} \not\sim \psi(\underline{x})$ for every term ψ with at least one occurrence of $\underline{x}$.

Structures satisfying the axioms (I) are called locally free algebras over the $\underline{O}_0, \ldots, \underline{O}_{p-1}$ is denoted by $\mathcal{L}$. Mal'cev showed that $ET(\mathcal{L})$ is decidable. On the other hand, Tarski, see [24], has established:

If $n_i \ge 2$ for at least one i, then $WS(\mathcal{L})$ is undecidable.

In this section we shall consider $WS(\mathcal{L})$ under the assumption $n_i = 1$ for each i; i.e., $\mathcal{L}$ is the class of locally free algebras over p unary operations. Hereafter all the operations O_i are assumed to be unary; to emphasize this, we will use the symbols S_i instead of O_i. Moreover, we shall assume p = 2; this is done merely for notational convenience, and the reader will encounter no difficulty should he wish to undertake the tedious job of revising our theorems and proofs so as to apply to arbitrary finite p. The monadic second-order language with

just the two unary operation symbols $\underline{S}_0$, $\underline{S}_1$ is denoted simply by L. Under

these assumptions, $\underline{\mathcal{L}}$ becomes the class of nonempty structures of type $\langle\langle 1,1\rangle\rangle$

satisfying the axioms

(II-1) $\underline{S}_0 \, \underline{x} \not\approx \underline{S}_1 \, \underline{y}$,

(II-2) $\underline{S}_0 \, \underline{x} \approx \underline{S}_0 \, \underline{y} \vee \underline{S}_1 \, \underline{x} \approx \underline{S}_1 \, \underline{y} \rightarrow \underline{x} \approx \underline{y}$,

(II-3) $\underline{x} \not\approx \psi(\underline{x})$ for every term ψ with at least one occurrence of $\underline{x}$.

We shall see later that the schema (II-3) can be replaced by a single weak

second-order axiom.

Let χ be the class of structures satisfying (II-1), (II-2) alone. The

elements of χ will be called χ-algebras. We shall show that WS(χ) is decidable,

and obtain the decidability of WS($\underline{\mathcal{L}}$) as a corollary of this result. Our first

step will be to conduct a mathematical analysis of the structures in the class

χ. This analysis will be used in subsequent metamathematical arguments to

reach the desired goals.

DEFINITION 5.1. A χ-algebra $\mathfrak{U}$ is simple if for every x, y $\in |\mathfrak{U}|$ there exists a

z such that x, y $\in \mathfrak{S}_{\mathfrak{U}}(z)$; if in addition there exists an element z such that

$\mathfrak{U} = \mathfrak{S}_{\mathfrak{U}}(z)$, then $\mathfrak{U}$ is generated and z is called a generator; if there is no such

z, then $\mathfrak{U}$ is ungenerated. The class of simple χ-algebras is denoted by χ_s,

and its subclasses of generated and ungenerated algebras by χ_g and χ_u,

respectively.

The term "generated" and "ungenerated" apply only to simple χ-algebras.

To avert confusion, however, we sometimes redundantly refer to χ-algebras as

"generated simple" or "ungenerated simple."

THEOREM 5.2. An algebra is a $\aleph$-algebra if and only if it is a cardinal sum of simple $\aleph$-algebras.

PROOF. That a cardinal sum of $\aleph$-algebras is again a $\aleph$-algebra is immediate from axioms (II-1), (II-2). Now let $\mathfrak{A} = \langle A, S_0, S_1 \rangle \in \aleph$; for x, y $\in$ A, we write $x \sim y$ if there exists $z \in A$ such that both x, y are members of $|\mathfrak{S}_{\mathfrak{A}}(z)|$. Clearly, $\sim$ is an equivalence relation. For any $x \in A$, let $\bar{x}$ be the equivalence class of x, and let $\bar{A} = \{\bar{x} : x \in A\}$. If, for any $\bar{x} \in \bar{A}$, we let $\mathfrak{A}_{\bar{x}} = \langle \bar{x}, S_0', S_1' \rangle$ where S_0', S_1' are the restrictions of S_0, S_1 to $\bar{x}$, then $\mathfrak{A}$ is the cardinal sum of the $\mathfrak{A}_{\bar{x}}$, $\bar{x} \in \bar{A}$.

Henceforth, when we say that ψ is a term, we mean that ψ is a term in the language L. The rank of ψ is denoted by $|\psi|$. The composition of two terms $\psi = \underline{S}_{i_0} \cdots \underline{S}_{i_{n-1}}(\underline{x})$, $\chi = \underline{S}_{j_0} \cdots \underline{S}_{j_{m-1}}(\underline{y})$ is denoted simply by concatenation: $\psi\chi$ is $\underline{S}_{i_0} \cdots \underline{S}_{i_{n-1}} \underline{S}_{j_0} \cdots \underline{S}_{j_{m-1}}(\underline{y})$. We say that ψ is a prefix of $\psi\chi$ and χ is a suffix of $\psi\chi$. ψ^1 is ψ itself, and for finite $n \geq 1$, $\psi^{n+1} = \psi^n\psi$. Given a structure $\mathfrak{A} = \langle A, S_0, S_1 \rangle$ and x, y $\in$ A, we write $x = \psi(y)$ just in case $F(x,y)$ holds in $\mathfrak{A}$, where $F(\underline{x},\underline{y})$ is the formula $\underline{x} \approx \psi(\underline{y})$.

LEMMA 5.3 (Cancellation Law). Let φ, χ, ψ be any terms.

(i) $\forall \underline{x} \forall \underline{y}[\varphi\chi(\underline{x}) \approx \varphi\psi(\underline{y}) \to \chi(\underline{x}) \approx \psi(\underline{y})]$ is true in $\aleph$.

(ii) If ψ has a prefix ψ' of the same rank as χ and $\psi = \psi'\psi''$, then

$\forall \underline{x} \forall \underline{y}[\chi(\underline{x}) \approx \psi(\underline{y}) \to \underline{x} \approx \psi''(\underline{y})]$ is true in $\aleph$.

PROOF: From (II-1), (II-2) by induction on the rank of the term φ in (i) and χ in (ii).

LEMMA 5.4. Let $\mathfrak{U}$ be a generated simple K-algebra. If for some $u \in |\mathfrak{U}|$ and some term ψ of L, $|\psi| > 0$ and $u = \psi(u)$ then

(i) u is a generator of $\mathfrak{U}$;

(ii) if $x \in |\mathfrak{U}|$ and $x = \chi(x)$ for some nontrivial term χ of L, then $x = \psi'(u)$ for some suffix ψ' of ψ.

PROOF. Let y be any generator of $\mathfrak{U}$, and let φ be a term such that $u = \varphi(y)$. There is a finite n such that $|\psi^n| \geq |\varphi|$. Now, $\psi^n(u) = \varphi(y)$ also, so by 5.3 (ii), $y = \psi'(u)$ where ψ' is a suffix of ψ^n. It follows that u is also a generator of $\mathfrak{U}$.

Now consider part (ii). From (i) we have that both u, x are generators of $\mathfrak{U}$. Say $u = \varphi(x)$; we then have $\psi^n(u) = \varphi(x)$ for every n. If $|\varphi| \leq |\psi|$, then we take $n = 1$ and have, by the cancellation law, $x = \psi'(u)$ for some suffix ψ' of ψ. If $|\varphi| > |\psi|$, let $n > 1$ be such that $|\psi^{n-1}| < |\varphi| \leq |\psi^n|$. Applying the cancellation law to the equation $\psi^n(u) = \varphi(x)$, we again obtain that $x = \psi'(u)$ for some suffix ψ' of ψ.

Figure 2 is a tree diagram of an element $\mathfrak{U} = \langle A, S_0, S_1 \rangle$ of K. Here the universe is the set $A = \{\epsilon, b\} \cup (\{a, ba\} \cdot \{a, b\}^*)$, and the two operations are, for all $x \in A$, $S_0(x) = xa$, $S_1(x) = xb$ if $x \neq b$, and $S_1(b) = \epsilon$. The structure $\mathfrak{U}$ shown satisfies the equation $S_1 S_1(\epsilon) = \epsilon$.
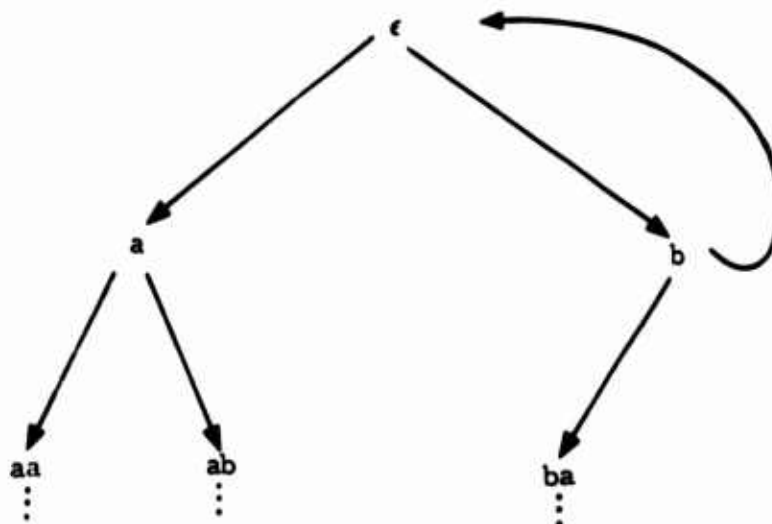
Figure 2. Diagram of a K-Algebra.

If a generated K-algebra also satisfies the axioms (II-3) (i.e., it fails to satisfy the hypothesis of 5.4), then it is isomorphic to $\mathfrak{R}_2$. Thus, the impact of Lemma 5.4 is that a generated K-algebra is either isomorphic to $\mathfrak{R}_2$ or has just one "loop." This is expressed in the following:

THEOREM 5.5. Two generated K-algebras $\mathfrak{A}$, $\mathfrak{B}$ are isomorphic if and only if either

(i)   each of $\mathfrak{A}$, $\mathfrak{B}$ is isomorphic to $\mathfrak{R}_2$

or else

(ii)   there exists a nontrivial term ψ such that $\exists \underline{x}[\underline{x} \approx \psi(\underline{x})]$ holds in both $\mathfrak{A}$, $\mathfrak{B}$, while for any nontrivial proper suffix ψ' of ψ, $\exists \underline{x}[\underline{x} \approx \psi'(\underline{x})]$ fails in both $\mathfrak{A}$, $\mathfrak{B}$.

Notice the one-to-one correspondence between terms and words over $\{a,b\}$: a term ψ corresponds to its value $\psi(\epsilon)$ in the particular K-algebra $\mathfrak{R}_2$. We thus

establish a many-to-one correspondence between words over $\{a,b\}$ and the isomorphism classes of generated κ-algebras; the empty word corresponds to the class of algebras isomorphic to $\mathfrak{N}_2$, and the other classes are determined by words distinct from ϵ: given an isomorphism class not containing $\mathfrak{N}_2$, let $\psi = \underline{S}_{i_0} \ldots, \underline{S}_{i_n}(\underline{x})$ be a term such that the condition of 5.5 (ii) is satisfied by every structure in the class; then the word $\psi(\epsilon) \in \{a,b\}^*$ corresponds to this class. Two words u, v determine the same isomorphism class if and only if there exist words u', u'' such that $u = u'u''$ and $v = u''u'$.

     Let $pred(\underline{x},\underline{y})$ be the formula

$$\underline{S}_0(\underline{x}) \approx \underline{y} \vee \underline{S}_1(\underline{x}) \approx \underline{y}.$$

If $y \in |\mathfrak{A}|$ for some κ-algebra $\mathfrak{A}$, there is at most one x such that $pred(x,y)$ holds in $\mathfrak{A}$ (although there may not be any such x, e.g., if $\mathfrak{A} = \mathfrak{N}_2$ and $y = \epsilon$). If there is an x such that $pred(x,y)$ we denote it by $pd(y)$; otherwise, we let $pd(y) = y$. We now put

$$pd^0(y) = y$$

$$pd^{n+1}(y) = pd(pd^n(y))$$

for each finite n.

DEFINITION 5.6. Let $\mathfrak{A}$ be any ungenerated simple κ-algebra. A <u>descriptor</u> of $\mathfrak{A}$ is any set M of natural numbers such that for some $x \in |\mathfrak{A}|$,

$M = M_{\mathfrak{A},x} = \{n : pd^n(x) = \underline{S}_0(pd^{n+1}(x))\}$.

THEOREM 5.7. Let $\mathfrak{A}$, $\mathfrak{B}$ be any ungenerated simple κ-algebra.

     (1)   If $x \in |\mathfrak{A}|$, then $\mathfrak{S}_{\mathfrak{A}}(x) \cong \mathfrak{N}_2$.

(ii) If x, $y \in |\mathfrak{A}|$ then there exist integers m, n such that, for each k, $m + k \in M_{\mathfrak{A},x}$ if and only if $n + k \in M_{\mathfrak{A},y}$.

(iii) $\mathfrak{A} \cong \mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have identical desciptors, i.e., there exist $x \in |\mathfrak{A}|$, $y \in |\mathfrak{B}|$ such that $M_{\mathfrak{A},x} = M_{\mathfrak{B},y}$.

PROOF. If $x \in |\mathfrak{A}|$, then $\mathfrak{S}_{\mathfrak{A}}(x)$ is a generated simple χ algebra. Suppose $\mathfrak{S}_{\mathfrak{A}}(x)$ is not isomorphic to $\mathfrak{N}_2$. Then by 5.5 (ii), there is a nontrivial term ψ and a $u \in |\mathfrak{S}_{\mathfrak{A}}(x)|$ such that $u = \psi(u)$. Let $y \in |\mathfrak{A}|$; then for some $z \in |\mathfrak{A}|$, u, $y \in |\mathfrak{S}_{\mathfrak{A}}(z)|$. Since $u = \psi(u)$, we have that u is a generator of $\mathfrak{S}_{\mathfrak{A}}(z)$ by 5.4 (i); hence $y \in |\mathfrak{S}_{\mathfrak{A}}(u)|$. Since y is arbitrary in $|\mathfrak{A}|$, it follows that $|\mathfrak{A}| \subseteq \mathfrak{S}_{\mathfrak{A}}(u)$. But this contradicts the hypothesis that $\mathfrak{A}$ is ungenerated.

If x, $y \in |\mathfrak{A}|$, let z be such that x, $y \in \mathfrak{S}_{\mathfrak{A}}(z)$, say $z = pd^n(y)$. Then $pd^{m+k}(x) = pd^{n+k}(y)$ for every k; hence, $m + k \in M_{\mathfrak{A},y}$, for $k = 0,1. \ldots$.

Finally, we consider (iii). The "only if" part is obvious. Let $x \in |\mathfrak{A}|$, $y \in |\mathfrak{B}|$ be such that $M_{\mathfrak{A},x} = M_{\mathfrak{B},y}$. Now $|\mathfrak{A}|$ consists of the following disjoint subsets

$$A_x = |\mathfrak{S}_{\mathfrak{A}}(x)|,$$

$$B_x = \{pd^n(x) : n < \omega\},$$

and for each n,

$$C_{x,n} = |\mathfrak{S}_{\mathfrak{A}}(\underline{S}_1(pd^{n+1}(x)))| \quad \text{if } n \in M_{\mathfrak{A},x},$$

$$= |\mathfrak{S}_{\mathfrak{A}}(\underline{S}_0(pd^{n+1}(x)))| \quad \text{otherwise.}$$

Similarly, $|\mathfrak{B}|$ consists of disjoint classes A_y, B_y, $C_{y,n}$, $n = 0, 1, 2, \ldots$.
We then have $\mathfrak{S}_{\mathfrak{A}}(A_x) \cong \mathfrak{S}_{\mathfrak{B}}(A_y)$ and $\mathfrak{S}_{\mathfrak{A}}(C_{x,n}) \cong \mathfrak{S}_{\mathfrak{B}}(C_{y,n})$, $n = 0, 1, \ldots$, since,
by (1), all of these structures are isomorphic to $\mathfrak{N}_2$. Let the function
$f : |\mathfrak{A}| \to |\mathfrak{B}|$ be defined as follows. On A_x, f is the natural isomorphism mapping
A_x to A_y, and, similarly, on each $C_{x,n}$, f is the natural isormorphism mapping
$C_{x,n}$ to $C_{n,y}$, $n = 0, 1, \ldots$. On B_x, we put

$$f(pd^n(x)) = pd^n(y), \quad n = 1, 2, \ldots \ .$$

That f is, in fact, an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ now follows from $M_{\mathfrak{A},x} = M_{\mathfrak{B},y}$.

Figure 3 is a tree-like diagram of an ungenerated $\mathcal{K}$-algebra. The
descriptor associated with the element x indicated is $\{0,3,6,\ldots\}$.

Theorem 5.7 shows that each ungenerated $\mathcal{K}$-algebra is determined up to
isomorphism by a single subset of the natural numbers. Two such sets lead to
the same algebra if and only if "ultimately, one is a translation of the
other," in the sense of 5.7 (ii).

Theorems 5.2, 5.5, and 5.7 provide a comprehensive analytic description
of the $\mathcal{K}$-algebras. We now turn to the application of this description to the
decision problem for $WS(\mathcal{K})$.

LEMMA 5.8. Each of the theories $WS(\mathcal{K}_s)$, $WS(\mathcal{K}_g)$, $WS(\mathcal{K}_u)$, and $WS(\mathfrak{L})$ is finitely
(semantically) axiomatizable.

PROOF. The required axioms will be formulated with the aid of some special
formulas (in addition to those already defined, e.g., pred $(\underline{x},\underline{y})$):

$$Clpd(\underline{x},\underline{X}) \equiv \forall \underline{u}[\underline{u} \not\equiv \underline{x} \wedge \underline{u} \in \underline{X} \to \forall \underline{v}[pred(\underline{v},\underline{u}) \to \underline{v} \in \underline{X}]]$$

$$Sal(\underline{x},\underline{y}) \equiv \exists \underline{X}[\underline{y} \in \underline{X} \wedge Clpd(\underline{x},\underline{X})] \wedge \forall \underline{X}[\underline{y} \in \underline{X} \wedge Clpd(\underline{x},\underline{X}) \to \underline{x} \in \underline{X}].$$
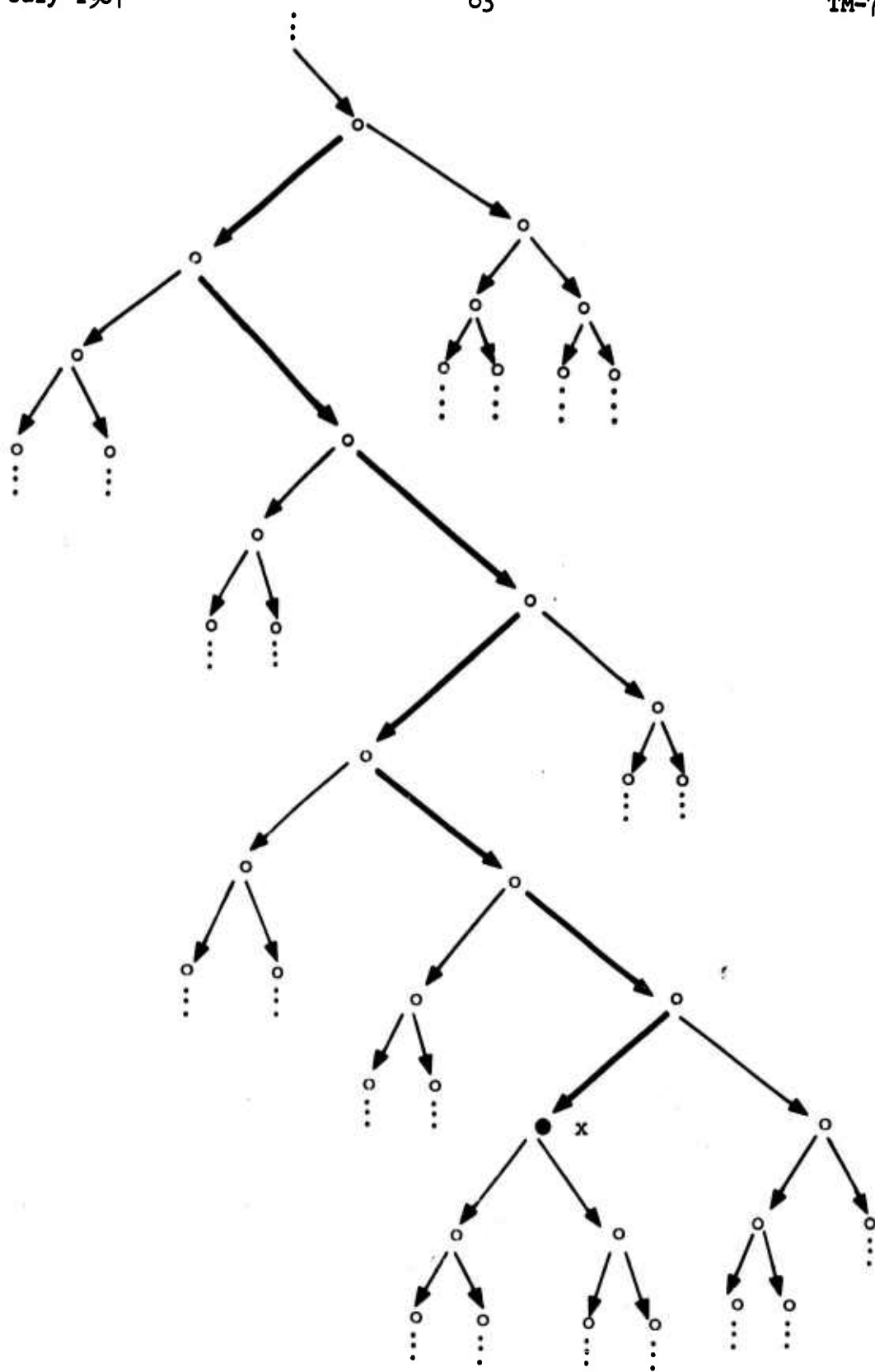
Figure 3.   Diagram of an Ungenerated X-Algebra.

Thus, $\text{Clpd}(x,X)$ holds in a $\mathcal{K}$-algebra $\mathfrak{A}$ just in case the predecessor operation, pd, maps $X \sim \{x\}$ into X. An elementary argument shows that for any x, $y \in |\mathfrak{A}|$, $\text{Sal}(x,y)$ if and only if $y \in \mathfrak{S}_{\mathfrak{A}}(x)$. (Notice that this equivalence remains true under the strong interpretation.) From this it follows that the class of $\mathcal{K}$-algebras satisfying

(1) $\forall \underline{x} \, \forall \underline{y} \, \exists \underline{z} [\text{Sal}(\underline{z},\underline{x}) \wedge \text{Sal}(\underline{z},\underline{y})]$

coincides with $\mathcal{K}_s$, the class of simple $\mathcal{K}$-algebras. The subclasses $\mathcal{K}_g$, $\mathcal{K}_u$, of $\mathcal{K}_s$ are, respectively, characterized by the additional axiom

(2) $\exists \underline{x} \, \forall \underline{y} \, \text{Sal}(\underline{x},\underline{y})$,

or its negation. Finally, we note that the axiom schema (II-3) is equivalent to the single weak second-order sentence

(II-4) $\neg \exists \underline{x} \, \exists \underline{y} [\underline{x} \not\approx \underline{y} \wedge \text{Sal}(\underline{x},\underline{y}) \wedge \text{Sal}(\underline{y},\underline{x})]$,

so that the class $\mathcal{L}$ of locally free algebras is determined by the axioms (II-1), (II-2), (II-4).

Theorem 5.5 and the remarks following it indicate a correspondence between words over $\{a,b\}$, i.e., elements of N_2 and the isomorphism classes of generated $\mathcal{K}$-algebras. Our proof of the following theorem is based on an implementation of this correspondence in the weak second-order language. Namely, we exhibit weak second-order formulas which, relative to any word u in $|\mathfrak{N}_2|$, define in $\mathfrak{N}_2$ a simple generated $\mathcal{K}$-algebra belonging to the isomorphism class corresponding to u.

THEOREM 5.9. $\text{WS}(\mathcal{K}_g)$ is decidable.

PROOF. Let

$$F(\underline{u},\underline{x}) \quad \equiv \neg\ \text{Sal}(\underline{u},\underline{x}) \ \vee \neg\ \exists\ \underline{z}\ \text{pred}(\underline{z},\underline{u})$$

$$G_0(\underline{u},\underline{x},\underline{y}) \equiv F(\underline{u},\underline{x}) \wedge F(\underline{u},\underline{y}) \wedge [(\underline{S}_0(\underline{x}) \not\approx \underline{u} \wedge \underline{y} \approx \underline{S}_0(\underline{x}))$$

$$\vee\ (\underline{S}_0(\underline{x}) \approx \underline{u} \wedge \neg\ \exists\ \underline{z}\ \text{pred}(\underline{z},\underline{y}))]$$

$$G_1(\underline{u},\underline{x},\underline{y}) \equiv F(\underline{u},\underline{x}) \wedge F(\underline{u},\underline{y}) \wedge [(\underline{S}_1(\underline{x}) \not\approx \underline{u} \wedge \underline{y} \approx \underline{S}_1(\underline{x}))$$

$$\vee\ (\underline{S}_1(\underline{x}) \approx \underline{u} \wedge \neg\ \exists\ \underline{z}\ \text{pred}(\underline{z},\underline{y}))].$$

(Note that $\neg\ \exists\ \underline{z}\ \text{pred}(\underline{z},y)$ holds for $y \in N_2$ iff $y = \epsilon$). Now let $u \in N_2$ and consider the structure $\mathfrak{A}(u) = \langle D, P_0, P_1 \rangle$, where

$$D = \{x : F(u,x)\},$$
$$P_0(x) = y \text{ iff } G_0(u,x,y),$$
$$P_1(x) = y \text{ iff } G_1(u,x,y).$$

If $u = \epsilon$, then $\mathfrak{A}(u)$ is $\mathfrak{N}_2$ itself. If $u \neq \epsilon$, say $u = \psi(\epsilon)$ for some nontrivial term ψ, then $\mathfrak{A}(u)$ is a generated K-algebra satisfying the axiom $\exists\ \underline{x}[\underline{x} \approx \psi(\underline{x})]$. It follows that every generated K-algebra is isomorphic to some $\mathfrak{A}(u)$, $u \in N_2$. Let E be any sentence in the language L. Using standard techniques of replacement of atomic formulas by formulas and relativization of quantifiers (e.g., a formula $\underline{S}_0(\underline{x}) \approx \underline{y}$ is replaced by $G_0(\underline{u},\underline{x},\underline{y})$, a quantifier $\exists\ \underline{X}\ \ldots$ is replaced by $\exists\ \underline{X}[\forall\ \underline{z}[\underline{z} \in \underline{X} \rightarrow F(\underline{u},\underline{z})] \wedge \ldots$, and so on), we can effectively obtain a formula $E'(\underline{u})$ such that E is true in $\mathfrak{A}(u)$ if and only if $E'(u)$ holds in $\mathfrak{N}_2$. Since every element of K_g is isomorphic to some $\mathfrak{A}(u)$, it follows that E is true in K_g if and only if $\forall\ \underline{u}\ E'(\underline{u})$ is true in $\mathfrak{N}_2$. But $WS(\mathfrak{N}_2)$ is decidable; hence, so is $WS(K_g)$.

Our next main result will be that $WS(\aleph_u)$ is decidable. The principal tools we shall use in the proof of this are the generalized products of Fererman and Vaught [12]. Since their results on generalized products apply only to elementary theories, we are obliged to replace $WS(\aleph_u)$ by an equivalent elementary theory: In fact, we shall correlate with each structure $\mathfrak{A}$ a structure $\mathfrak{A}^+$ such that $WS(\mathfrak{A})$ is decidable iff $ET(\mathfrak{A}^+)$ is decidable.

<u>DEFINITION 5.10.</u> Let $\mathfrak{A} = \langle A, O_0, \ldots, O_{m-1}, R_0, \ldots, R_{n-1} \rangle$ be an algebraic structure, and let A^+ be the set of all finite subsets of A. Then

$$\mathfrak{A}^+ = \langle A \cup A^+, A, e, O'_c, \ldots, O'_{n-1}, R_0, \ldots, R_{n-1} \rangle$$

where each O'_i is a m_i+1-ary relation such that $O'_i(x_0, \ldots, x_{n_i}, y)$ iff $O_i(x_0, \ldots, x_{n_i}) = y$, and $e(x,y)$ holds between two elements x, y of $A \sqcup A^+$ if and only if $x \in A$, $y \in A^+$, and $x \in y$.

<u>LEMMA 5.11.</u> Let $\mathfrak{A}$ be any algebraic structure. We can effectively correlate with each formula $F(\underline{x}_0, \ldots, \underline{x}_{m-1}, \underline{X}_0, \ldots, \underline{X}_{n-1})$ in the monadic second-order language of $\mathfrak{A}$ a formula $F'(\underline{x}_0, \ldots, \underline{x}_{m-1}, \underline{y}_0, \ldots, \underline{y}_{n-1})$ in the elementary language of $\mathfrak{A}^+$ such that, for any $x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1} \in |\mathfrak{A}^+|$, $F'(x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1})$ holds in $\mathfrak{A}^+$ if and only if

(i) $x_0, \ldots, x_{m-1} \in |\mathfrak{A}|$,

(ii) $y_0, \ldots, y_{n-1} \in |\mathfrak{A}|^+$,

and

(iii) $F(x_0, \ldots, x_{m-1}, y_0, \ldots, y_{n-1})$ holds in $\mathfrak{A}$.

PROOF: by standard techniques of eliminating terms in favor of relation symbols, replacement of atomic formulas, and relativization of qualifiers.

A given ungenerated χ-algebra $\mathfrak{A}$ is determined up to isomorphism by any of its descriptors. The following definition and lemma provide an explicit method for the construction of an ungenerated χ-algebra with any given descriptor. Some of the technical features of this construction facilitate a later argument involving generalized products.

DEFINITION 5.12. Let U be any subset of the natural numbers. Then $\mathfrak{B}(U)$ is the relational structure $\langle B \cup B', B, E, T_o, T_1 \rangle$ where

(i)   B is the set of all sequences $f = \langle f(0), f(1), \ldots \rangle$ such that $f(i) = \emptyset$ except for one i, denoted by $\hat{f}$, and $f(\hat{f}) \in N_2$, subject to the restriction that, if $\hat{f} > 0$, then $f(\hat{f}) \in \{b\} \cdot N_2$ only if $\hat{f}-1 \in U$ and $f(\hat{f}) \in \{a\} \cdot N_2$ only if $\hat{f}-1 \notin U$;

(ii)   B' is the set of all sequences $f = \langle f(0), f(1), \ldots \rangle$ such that $f(i)$ is a finite subset of N_2 for each i, $f(i) \neq \emptyset$ for at most finitely many i, and for each i and $w \in N_2$, $w \in f(i)$ only if the sequence g such that

$$g(j) = \emptyset \text{ if } j \neq i,$$
$$= w \text{ if } j = i,$$

is a member of B;

(iii) E(f,g) holds if and only if $f \in B$, $g \in B'$, and $f(\hat{f}) \in g(\hat{f})$;

(iv) $T_o(f,g)$ is defined only for f, $g \in B$: in case

$$\hat{f} > 0, \quad \hat{f}-1 \in U, \text{ and } f(\hat{f}) = \epsilon,$$

then $T_0(f,g)$ holds if and only if

$$\hat{g} = \hat{f}{-}1 \text{ and } g(\hat{g}) = \epsilon,$$

and in all other cases, $T_0(f,g)$ holds if and only if

$$\hat{g} = \hat{f} \text{ and } g(\hat{g}) = f(\hat{f})a;$$

(v) $T_1(f,g)$ is defined only for $f, g \in B$:   in case

$$\hat{f} > 0, \ \hat{f}{-}1 \notin U, \text{ and } f(\hat{f}) = \epsilon,$$

then $T_1(f,g)$ holds if and only if

$$\hat{g} = \hat{f}{-}1 \text{ and } g(\hat{g}) = \epsilon,$$

and in all other cases, $T_1(f,g)$ holds if and only if

$$g = f \text{ and } g(g) = f(f)b.$$

LEMMA 5.13. Let U be any set of natural numbers. Then there exist an un-generated $\aleph$-algebra $\mathfrak{A}$ such that $\mathfrak{A}^+ \simeq \mathfrak{B}(U)$; moreover, U is a descriptor of $\mathfrak{A}$.

PROOF. Let $\mathfrak{B}(U)$ be as in definition 5.12, and put $\mathfrak{A} = \langle B, S_0, S_1 \rangle$, where for each $f \in B$, $S_0(f)$ is the unique $g \in B$ such that $T_0(f,g)$, and S_1 is defined analogously from T_1. The verifications that S_0, S_1 are well-defined, that $\mathfrak{A}^+ \simeq \mathfrak{B}(U)$, and that U is a descriptor of $\mathfrak{A}$ are purely routine.

Figure 4 is a diagram of a structure $\mathfrak{B}(U)$ where $U = \{0,3,\ldots\}$. Each node corresponds to a distinct sequence f in B, and the value of this sequence at the one place where it is different from $\emptyset$ is indicated alongside.

We shall briefly summarize the definitions and theorems concerning generalized products which are required for the proof of the decidability of $\mathrm{ET}(\aleph_u^+)$. All of this material is drawn from [12], with some minor changes,
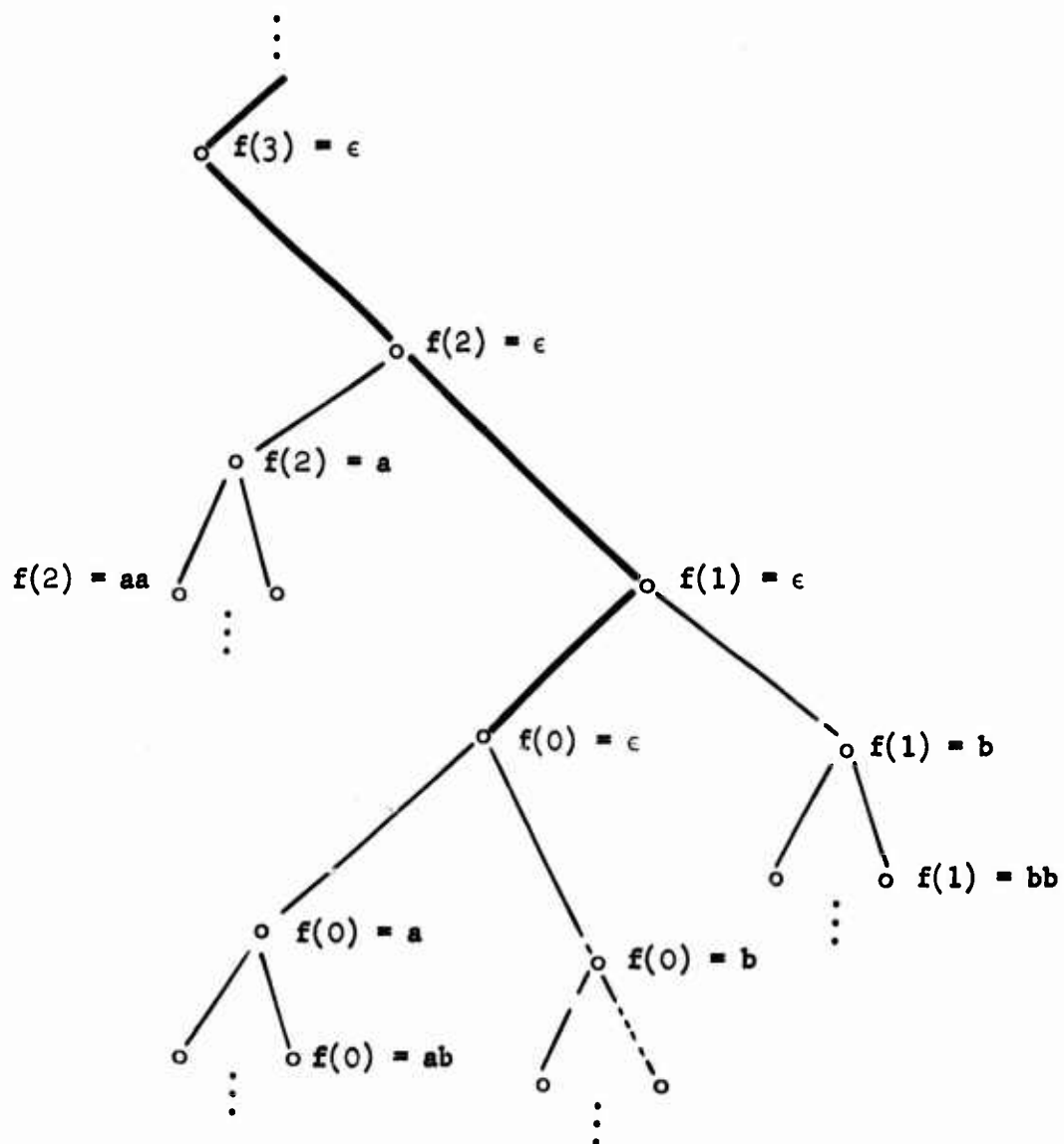
Figure 4. Diagram of a Structure $\mathfrak{B}(U)$ Where $U = \{0,3,\ldots\}$.

mostly notational. The reader should have little difficulty in reconciling
these differences. (The principal difference is our use of monadic second-
order theories instead of the somewhat more general "subset algebras.")

Let $\mathfrak{A}^{(i)}$, $i \in I$, be a set of relational structures of the same similarity
type, indexed by members of the nonempty set I. Let $\mathfrak{S}$ be any relational struc-
ture such that $|\mathfrak{S}| = I$. Let F be a formula in the monadic second-order language
of $\mathfrak{S}$, and let $G_0, \ldots, G_m$ be elementary formulas in the language of the $\mathfrak{A}^{(i)}$. The
sequence $\zeta = \langle F, G_0, \ldots, G_m \rangle$ is called a <u>standard acceptable sequence with free</u>
<u>variables</u> $\underline{x}_0, \ldots, \underline{x}_n$ if

(i)   The free variables of F are at most the set variables $\underline{X}_0, \ldots, \underline{X}_m$;

(ii) a variable occurs free in some G_i if and only if it is one of
$\underline{x}_0, \ldots, \underline{x}_n$.

Let D be the set of all functions $f : I \to \bigcup_{i \in I} |\mathfrak{A}^{(i)}|$ such that $f(i) \in |\mathfrak{A}^{(i)}|$
for each $i \in I$. A standard acceptable sequence $\zeta = \langle F, G_0, \ldots, G_m \rangle$ with one
free variable $\underline{x}_0$ defines a set $D' \subseteq D$ if

$D' = \{f : F(X_0, \ldots, X_m)$ holds in $\mathfrak{S}$ (under the strong interpretation)$\}$
where for $j = 0, \ldots, m$,

$X_j = \{i : i \in I$ and $G_j(f(i))$ holds in $\mathfrak{A}^{(i)}$ (under the strong interpretation)$\}$

In case the sequence ζ has free variables $\underline{x}_0, \ldots, \underline{x}_n$, $n > 0$, we say that ζ
defines a n-ary relation on D, with an analogous meaning. Finally, a <u>relativized</u>

generalized product of the $\mathfrak{U}^{(i)}$ with respect to $\mathfrak{S}$ is a relational structure $\langle D', R_0', \ldots, R_{p-1}' \rangle$ where

    (i) D' is defined by a standard acceptable sequence with one free variable;

    (ii) each R_j', $j < p$, is obtained by restricting a relation R_j on D to D' where R_j is defined by a standard acceptable sequence.

Thus, each series of standard acceptable sequences $\zeta_0, \ldots, \zeta_n$ such that ζ_0 has exactly one free variable defines a relativized generalized product $\rho(\mathfrak{U}, \mathfrak{S})$. (In this notation, $\mathfrak{U}$ denotes the entire sequence of $\mathfrak{U}^{(i)}$, $i \in I$; i.e., $\mathfrak{U}$ is a function with domain I such that each value is a structure of a given similarity type.) If α, $\mathfrak{g}$ are classes of relational structures (of suitable similarity types) then $\rho(\alpha, \mathfrak{g})$ is the class of all products $\rho(\mathfrak{U}, \mathfrak{S})$ where $\mathfrak{S} \in \mathfrak{g}$ and for each $i \in |\mathfrak{S}|$, $\mathfrak{U}^{(i)} \in \alpha$. The basic theorem on generalized products (Theorem 3.1, [12] states:

> Any set or relation definable in the elementary theory
> of a relativized generalized product can also be defined
> by a standard acceptable sequence; moreover, this sequence
> can be effectively obtained from the defining formula.

We shall not use this theorem directly, but rather the following consequence of it (see Theorem 5.6, [12]):

> If ET(α) and SS($\mathfrak{g}$) are decidable, then so is ET($\rho(\alpha, \mathfrak{g})$).

THEOREM 5 14. Let $\mathfrak{g}$ be the class of all structures $\langle \omega, <, U \rangle$, where $U \subseteq \omega$. Then there i 1 relativized generalized product ρ such that $\rho(\{\mathfrak{M}_2^+\}, \mathfrak{g})$ is the class of all $\mathfrak{B}(U)$, $U \subseteq \omega$.

PROOF. We must exhibit a series of standard acceptable sequences ζ_0, ζ_1, ζ_2, ζ_3, ζ_4 defining the product ρ. The language of $\mathfrak{A}$ has the constant predicates $<$ and $\underline{U}$, and in the language of $\mathfrak{N}_2^+ = \langle N_2 \cup N_2^+, N_2, e, R_0, R_1 \rangle$, we use the symbols $\underline{N}$, $\underline{e}$, $\underline{R}_0$, $\underline{R}_1$. In both languages, unsubscripted variables appearing in a formula denote any variables (of the appropriate type) which do not occur free in the formula. The letters f, g denote functions from ω to $|\mathfrak{N}_2^+|$. We take $\mathfrak{S} = \langle \omega, <, U \rangle$ to be an arbitrary element of $\mathfrak{A}$, and we let $\mathfrak{B}(U) = \langle B \cup B', B, E, T_0, T_1 \rangle$ be as in Definition 5.12. Our definitions of the sequences $\zeta_0, \cdots, \zeta_4$ will be given in such a way that only a straightforward check against Definition 5.12 (which we omit) is required to verify that they do, in fact, define the universe and relations of $\mathfrak{B}(U)$.

Let

$$
\begin{aligned}
F_1(\underline{X}_0, \underline{X}_1, \underline{X}_2, \underline{X}_3) \equiv &\ \forall \underline{x}[\underline{x} \in \underline{X}_0 \rightarrow \underline{x} \in \underline{X}_1] \\
&\land \exists \underline{x}[\underline{x} \in \underline{X}_0 \land \forall \underline{y}[\underline{y} \in \underline{X}_0 \rightarrow \underline{x} \approx \underline{y}]] \\
&\land \forall \underline{x} \forall \underline{y}[\underline{x} \in \underline{X}_0 \land \underline{y} < \underline{x} \land \neg \exists \underline{z}[\underline{y} < \underline{z} \land \underline{z} < \underline{x}] \\
&\rightarrow (\underline{x} \in \underline{X}_2 \rightarrow \neg \underline{U}(\underline{y})) \land (\underline{x} \in \underline{X}_3 \rightarrow \underline{U}(\underline{y}))].
\end{aligned}
$$

Now if

$$
X_0 = \{n : f(n) \in N_2\},
$$

$$
X_1 = \{n : f(n) = \emptyset\},
$$

$$
X_2 = \{n : f(n) \in \{a\} \cdot N_2\},
$$

$$
X_3 = \{n : f(n) \in \{b\} \cdot N_2\},
$$

then $F_1(X_o, X_1, X_2, X_3)$ holds in $\mathfrak{S}$ (under the strong interpretation) if and only if f is a member of B. We need only define formulas G_j such that $X_j = \{n : G_j(f(n))$ holds in $\mathfrak{N}_2^+\}$, $j = 0, \ldots, 3$. Put

$$G_o(\underline{x}_o) \equiv \underline{N}(\underline{x}_o),$$

$$G_1(\underline{x}_o) \equiv \neg\, \underline{N}(\underline{x}_o) \wedge \neg\, \exists\, \underline{y}\ \underline{e}(\underline{y}, \underline{x}_o),$$

$$G_2(\underline{x}_o) \equiv \exists\, \underline{x}\, \exists\, \underline{y}[\underline{N}(\dot{\underline{x}}) \wedge \neg\, \exists\, \underline{z}[\underline{R}_o(\underline{z}, \underline{x}) \vee R_1(\underline{z}, \underline{x})] \wedge \underline{R}_o(\underline{x}, \underline{y})$$
$$\wedge\ \mathrm{Sal}'(\underline{y}, \underline{x}_o)],$$

where Sal' is obtained from Sal as in Lemma 5.11, and

$$G_3(\underline{x}_o) \equiv \exists\, \underline{x}\, \exists\, \underline{y}[\underline{N}(\underline{x}) \wedge \neg\, \exists\, \underline{z}[\underline{R}_o(\underline{z}, \underline{x}) \vee \underline{R}_1(\underline{z}, \underline{x})] \wedge \underline{R}_1(\underline{x}, \underline{y})$$
$$\wedge\ \mathrm{Sal}'(\underline{y}, \underline{x}_o)].$$

These formulas satisfy the required condition, and hence, $\zeta_1 = \langle F_1, G_o, G_1, G_2, G_3 \rangle$ is a standard acceptable sequence defining B.

The sequence ζ_o must define $B \cup B'$, the domain of $\mathfrak{B}(U)$. Let

$$F_o(\underline{X}_1, \underline{X}_4, \underline{X}_5, \underline{X}_6) \equiv \exists\, \underline{x}\, \forall\, \underline{y}[\underline{x} < \underline{y} \to \underline{y} \in \underline{X}_1] \wedge \forall\, \underline{x}[\underline{x} \in \underline{X}_4]$$
$$\wedge \forall\, \underline{x}\, \forall\, \underline{y}[\underline{y} < \underline{x} \wedge \neg\, \exists\, \underline{z}[\underline{y} < \underline{z} \wedge \underline{z} < \underline{x}]]$$
$$\to (\underline{x} \in \underline{X}_5 \wedge \neg\, \underline{U}(\underline{y})) \vee (\underline{x} \in \underline{X}_6 \wedge \underline{U}(\underline{y}))].$$

If now X_1 is as before, and

$$X_4 = \{n : f(n) \text{ is a finite subset of } N_2\}$$

$$X_5 = \{n : f(n) \subseteq (\{a\} \cdot N_2) \cup \{\epsilon\}\},$$

$$X_6 = \{n : f(n) \subseteq (\{b\} \cdot N_2) \cup \{\epsilon\}\},$$

then $F_0(X_1,X_4,X_5,X_6)$ holds in $\mathfrak{S}$ if and only if $f \in B'$. Thus, we let

$$G_4(\underline{x}_0) \equiv \neg\, \underline{N}(\underline{x}_0),$$

$$G_5(\underline{x}_0) \equiv \neg\, \underline{N}(\underline{x}_0) \wedge \forall\, \underline{x}[\underline{e}(\underline{x},\underline{x}_0) \rightarrow \neg\, \exists\, \underline{z}[\underline{R}_0(\underline{z},\underline{x}) \vee \underline{R}_1(\underline{z},\underline{x})] \vee G_2(\underline{x})],$$

$$G_6(\underline{x}_0) \equiv \neg\, \underline{N}(\underline{x}_0) \wedge \forall\, \underline{x}[\underline{e}(\underline{x},\underline{x}_0) \rightarrow \neg\, \exists\, \underline{z}[\underline{R}_0(\underline{z},\underline{x}) \vee \underline{R}_1(\underline{z},\underline{x})] \vee G_3(\underline{x})],$$

and find that

$$\zeta_0 = \langle F_0 \vee F_1, G_0, \ldots, G_6 \rangle$$

is a standard acceptable sequence defining $B \cup B'$.

The sequence ζ_2 should define the relation E of $\mathfrak{B}(U)$. Let

$$F_2(\underline{X}_0) \equiv \exists\, \underline{x}[\underline{x} \in \underline{X}_0];$$

then if

$$X_0 = \{n : f(n) \in g(n)\},$$

we find that, for $f, g \in B \cup B'$, $F_2(X_0)$ holds in $\mathfrak{S}$ if and only if $E(f,g)$. Put

$$G_7(\underline{x}_0,\underline{x}_1) \equiv e(\underline{x}_0,\underline{x}_1).$$

Then $\zeta_2 = \langle F_2, G_7 \rangle$ is a standard acceptable sequence, and E is the restriction to $B \cup B'$ of the relation defined by it.

Let

$$\begin{aligned}
F_3(\underline{X}_0,\underline{X}_1,\underline{X}_2,\underline{X}_3) \equiv\ &\exists\, \underline{x}\, \exists\, \underline{y}[\underline{x} \in \underline{X}_0 \wedge \underline{y} \in \underline{X}_1 \wedge \{(\exists\, \underline{x}'[\underline{x}' < \underline{x} \\
&\wedge \neg\, \exists\, \underline{z}[\underline{x}' < \underline{z} \wedge \underline{z} < \underline{x}] \wedge \underline{U}(\underline{x}') \wedge \underline{x} \in \underline{X}_2] \\
&\wedge \underline{y} < \underline{x} \wedge \neg\, \exists\, \underline{z}[\underline{y} < \underline{z} \wedge \underline{z} < \underline{x}] \wedge \underline{y} \in \underline{X}_3) \\
&\wedge (\neg\, \exists\, \underline{x}'[\underline{x}' < \underline{x} \wedge \neg\, \exists\, \underline{z}[\underline{x}' < \underline{z} \wedge \underline{z} < \underline{x}] \\
&\wedge \underline{U}(\underline{x}') \wedge \underline{x} \in \underline{X}_2] \wedge \underline{y} \approx \underline{x} \wedge \underline{y} \in \underline{X}_4)\}];
\end{aligned}$$

then if

$$X_0 = \{n : f(n) \in N_2\},$$

$$X_1 = \{n : g(n) \in N_2\},$$

$$X_2 = \{n : f(n) = \epsilon\},$$

$$X_3 = \{n : g(n) = \epsilon\},$$

$$X_4 = \{n : g(n) = f(n)a\},$$

we have that for $f, g \in B \cup B'$, $F(X_0, X_1, X_2, X_3, X_4)$ holds in $\mathfrak{S}$ iff $T_0(f,g)$. Thus, we put

$$G_8(\underline{x}_0) \quad \equiv \underline{N}(\underline{x}_0)$$

$$G_9(\underline{x}_1) \quad \equiv \underline{G}_8(\underline{x}_1)$$

$$G_{10}(\underline{x}_0) \quad \equiv \underline{N}(\underline{x}_0) \wedge \neg \exists \underline{z}[\underline{R}_0(\underline{z},\underline{x}_0) \vee \underline{R}_1[\underline{z},\underline{x}_0)]$$

$$G_{11}(\underline{x}_1) \quad \equiv G_{10}(\underline{x}_1)$$

$$G_{12}(\underline{x}_0,\underline{x}_1) \equiv \underline{R}_0(\underline{x}_0,\underline{x}_1),$$

and have that $\zeta_3 = \langle F_3, G_8, G_9, G_{10}, G_{11}, G_{12} \rangle$ is a standard acceptable sequence and the restriction to $B \cup B'$ of the relation defined by it coincides with T_0.

The sequence ζ_4 defining an extension of T_1 is similar to ζ_3, and is obtained from the latter by making suitable minor modifications to the formulas F_3 and G_{12}. This completes the proof of Theorem 5.14.

COROLLARY 5.15. $WS(\aleph_u)$ is decidable.

PROOF. The decidability of $WS(\aleph_u)$ is equivalent to the decidability of $ET(\aleph_u^+)$. By theorem 5.14 and the results of Feferman and Vaught, this, in turn, follows from the decidability of $WS(\mathfrak{N}_2^+)$, which we know from 3.8 and 5.11, and of $SS(\mathfrak{g})$,

where $\mathfrak{g}$ is the class of all $\langle \omega, <, U \rangle$, $U \subseteq \omega$. In [2], Buchi established the decidability of $SS(\langle \omega, < \rangle)$. Now a sentence F is in $SS'(\mathfrak{g})$ if and only if the sentence $\forall \underline{X}\, F'(\underline{X})$, where $F'(\underline{X})$ is obtained from F by replacing each occurence of the unary predicate symbol $\underline{U}$ by the set variable $\underline{X}$, is in $SS(\langle \omega, < \rangle)$. Thus, $SS(\mathfrak{g})$ is decidable, and hence, so is $WS(K_u)$.

COROLLARY 5.16. The weak second-order theory of simple K-algebras, $WS(K_s)$, is decidable.

PROOF: by 5.9 and 5.15.

Cardinal sums were included by Feferman and Vaught among their examples of relativized generalized products. We cannot directly use this result in a proof of the decidability of $WS(K)$, for the class K^+ is not the same as the class of cardinal sums of members of K_s^+. Nevertheless, we can still use generalized products to prove

THEOREM 5.17. $WS(K)$ is decidable.

PROOF. By the Lowenheim-Skolem theorem, as it applies to weak second-order theories, we may restrict the class K to contain only countable cardinal sums of members of K_s. Let $\mathfrak{g}$ be the class of all structures $\langle U, < \rangle$ where $\emptyset \neq U \subseteq \omega$ and $<$ is the order relation on natural numbers restricted to U. The decidability of $SS(\mathfrak{g})$ follows immediately from that of $SS(\langle \omega, < \rangle)$. Let $\mathfrak{S} = \langle u, < \rangle$ be a member of $\mathfrak{g}$, let $\mathfrak{A}^{(i)} = \langle A^{(i)}, S_0^{(i)}, S_1^{(i)} \rangle \in K_s$ for each $i \in U$, and let $\mathfrak{B}$ be the cardinal sum of the $\mathfrak{A}^{(i)}$. We wish to define a relativized generalized product ρ such that $\rho(\mathfrak{A}^+, \mathfrak{S}) \cong \mathfrak{B}^+$; actually, we shall only give an informal description

of $\rho(\mathfrak{A},\mathfrak{S})$, for the reader who has studied either [12] or the proof of 5.14

should have little difficulty in supplying the necessary standard acceptable

sequences. f, g denote functions with domain U such that for each i, $f(i)$,

$g(i) \in |\mathfrak{A}^{(i)+}|$.

The universe of $\rho(\mathfrak{A}^{+},)$ is $B \cup B'$, where

(i) $f \in B$ iff for some $i_o \in U$, $f(i_o) \in A^{(i_o)}$, while for all $i \neq i_o$,

$f(i) = \emptyset$;

(ii) $f \in B'$ iff for every $i \in U$, $f(i)$ is a finite subset of $A^{(i)}$, and

for only finitely many i do we have $f(i) \neq \emptyset$.

The relations of $\rho(\mathfrak{A}^{+},\mathfrak{S})$ are

(i) B,

(ii) $e(f,g)$, which holds iff $f \in B$, $g \in B'$, and for every $i \in U$, either

$f(i) = \emptyset$, or $f(i) \in g(i)$,

(iii) $R_o(f,g)$, which holds iff f, $g \in B$ and for every $i \in U$, either

$f(i) = g(i) = \emptyset$, or $R_o^{(i)}(f(i),g(i))$,

(iv) $R_1(f,g)$, analogous to R_o.

The product ρ, thus described, establishes the decidability of $ET(\mathcal{K}^{+})$, and

hence of $WS(\mathcal{K})$.

A similar proof of 5.17 which does not use the Lowenheim-Skolem theorem

can be given. However, this proof uses the most general form of the generalized

products, wherein the relational structure over the index set is replaced by a

subset algebra.

It is not difficult to show that Theorem 5.17 cannot be further improved; in the sense that neither of the Axioms II-1, II-2 can be omitted while retaining decidability.

COROLLARY 5.18. Let $\mathfrak{F}$, $\mathfrak{F}_\omega$, and $\mathfrak{F}_k$ for $k < \omega$ be the subclass of $\mathfrak{L}$ consisting of, respectively, the free algebras, the free algebras with infinitely many generators, and the free algebras with k generators. Then each of $WS(\mathfrak{F})$, $WS(\mathfrak{F}_\omega)$, $WS(\mathfrak{F}_k)$, $k < \omega$, and $WS(\mathfrak{L})$ is decidable.

PROOF: Each of these subtheories of $WS(X)$ is finitely (semantically) axiomatizable.

As remarked at the end of Section 4, M.O. Rabin has recently found a proof of the decidability of $SS(\mathfrak{R}_p)$ for every finite p.[*] The constructions and proofs of this section require only minor modifications to handle the strong second-order case. The proof of 5.17, at least, becomes simpler, and no use of the Lowenheim-Skolem theorem is required. Thus, with Rabin's result as a starting point, we can establish the decidability of $SS(X)$ and $SS(\mathfrak{L})$.

---

[*]See [22].

REFERENCES

[1] Buchi, J.R. "Weak Second-Order Arithmetic and Finite Automata," Zeitschr. f. Math. Logik und Grundlagen d. Math., Vol. 6 (1960), pp. 66-92.

[2] Buchi, J.R. "On a Decision Method in Restricted Second-Order Arithmetic," Logic, Method., and Phil. of Science, Proc. 1960 Int. Congress, Stanford University Press, 1962.

[3] Buchi, J.R. "Transfinite Automata Recursions and Weak Second-Order Theory, of Ordinals," Logic, Method., and Phil. of Science, Proc. 1964 Int. Congress, Amsterdam: North-Holland Publishing Co.

[4] Buchi, J.R. "Decision Methods in the Theory of Ordinals," Bull. Amer. Math. Soc., Vol. 71 (1965), pp. 767-770.

[5] Buchi, J.R. and C.C. Elgot: "Decision Problems of Weak Second-Order Arithmetic and Finite Automata," Abstract No. 553-112, Notices, Amer. Math. Soc., Vol. 5 (1958), p. 834.

[6] Chomsky, N. and G.A. Miller: "Finite State Languages," Inform. and Contr., Vol. 1 (1958), pp. 91-112.

[7] Doner, J.E. "Decidability of the Weak Second-Order Theory of Two Successors," Abstract No. 65T-468, Notices, Amer. Math. Soc., Vol. 12 (1965), p. 819.

[8] Doner, J.E. "Decidability of Locally Free Algebras with Unary Operations," Abstract No. 66T-384, Notices, Amer. Math. Soc., Vol. 13 (1966), pp. 634-635.

[9] Ehrenfeucht, A. "An Application of Games to the Completeness Problem for Formalized Theories," Fund. Math., Vol. 49 (1960/1961), pp. 129-141.

[10] Elgot, C.C. "Decision Problems of Finite Automata Design and Related Arithmetics," Trans. Amer. Math. Soc., Vol. 98 (1961), pp. 21-51.

[11] Eršov, Ju. L. "Decidability of Certain Non Elementary Theories," (Russian), Algebra i Logika Sem., Vol. 3 No. 2 (1965), pp. 45-47.

[12] Feferman, S. and R.L. Vaught: "The First-Order Properties of Algebraic Systems," Fund. Math., Vol. 47 (1959), pp. 57-103.

[13] Ginsburg, S. An Introduction to Mathematical Machine Theory. Reading, Mass.: Addison-Wesley, 1962.

REFERENCES (Continued)

[14] Ginsburg, S. The Mathematical Theory of Context-Free Languages. New York: McGraw-Hill, 1966.

[15] Ginsburg, S. and H.G. Rice. Two Families of Languages Related to ALGOL. J.Assoc. Comput. Mach., Vol. 9 (1962), pp. 350-371.

[16] Ginsburg, S. and G.F. Rose. "Operations Which Preserve Definability in Languages," J. Assoc. Comput. Mach., Vol. 10 (1963), pp. 175-195.

[17] Mal'cev, A.I. "Axiomatizable Classes of Locally Free Algebras of Certain Types," (Russian), Sibirsk. Mat. Z., Vol. 3 (1962), pp. 729-743.

[18] Medvedev, I.T. On a Class of Events Representable in a Finite Automaton (translated from the Russian by J. Schorr-Kon), M.I.T. Lincoln Laboratory Report 34-73, June 30, 1958.

[19] Mezei, J. and J.B. Wright. Generalized ALGOL-Like Languages. IBM Research Paper RC-1528, December 20, 1965.

[20] Myhill, J. "Finite Automata and the Representation of Events," in Wright Air Development Command Technical Report 57-624, 1957, pp. 112-137.

[21] Rabin, M.O. and D. Scott. "Finite Automata and Their Decision Problems," IBM J. Research and Develop., Vol. 3 (1959), pp. 114-125.

[22] Rabin, Michael O. "Decidability of Second-Order Theories and Automata on Infinite Trees," Trans. Amer. Math. Soc., Vol. 141 (1969), pp. 1-35.

[23] Robinson, R.M. "Restricted Set Theoretic Definitions in Arithmetic," Proc. Amer. Math. Soc., Vol. 9 (1958), pp. 238-242.

[24] Tarski, A. "Some Decision Problems for Locally Free Commutative Algebras," Abstract No. 66T-383, Notices, Amer. Math. Soc., Vol. 13, (1966), pp. 634.

[25] Tarski, A., A. Mostowski, and R.M. Robinson. Undecidable Theories, Amsterdam: North Holland Publishing Co., 1953.

[26] Thatcher, J.W. "A Further Generalization of Finite Automata and an Application to a Decision Problem," Abstract No. 67T-385, Notices, Amer. Math. Soc., Vol. 14 (1967), p. 534.

[27] Thatcher, J.W. and J.B. Wright. "Generalized Finite Automata, "Abstract No. 65T-649, Notices, Amer. Math. Soc., Vol. 12 (1965), p. 820.

[28] Thatcher, J.W. and J.B. Wright. "Generalized Finite Automata with an Application to a Decision Problem of Second-Order Logic," IBM Research Report RC-1713, November 16, 1966.

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13. ABSTRACT

This paper concerns a generalization of finite automata, the "tree acceptors," which have as their inputs finite trees of symbols rather than the usual sequences of symbols. Ordinary finite automata prove to be special cases of tree acceptors, and many of the results of finite automata theory continue to hold in their appropriately generalized forms. The tree acceptors provide new characterizations of the classes of regular sets and of context-free languages. The theory of tree acceptors is applied to a decision problem of mathematical logic. It is shown here that the weak second-order theory of two successors is decidable, thus settling a problem of Büchi. This result is in turn applied to obtain positive solutions to the decision problems for various other theories, e.g., the weak second-order theories of order types built up from the finite types, ω, and η, (the type of the rationals) by finitely many applications of the operations of order type addition, multiplication and converse; and the weak second-order theory of locally free algebras with only unary operations.

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